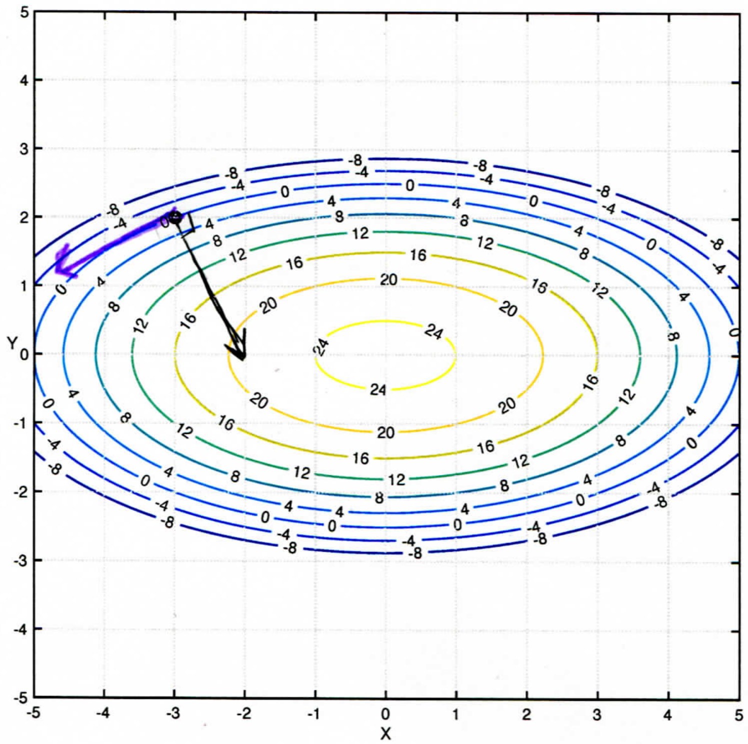
MATH 253 Name: SOLUTIONS

Quiz 6 Due at 5 pm on March 5, 2021

Instructions: (15 points total – 5pts. each) Show all work for credit. You may use your book, but no other resource. **GS:** Scan FOUR pages for your solutions.

1. Consider the function of two variables $f(x, y) = 25 - x^2 - 4y^2$ and its contour plot for various levels k.



Key:

Neither vector is drawn to scale, but they MUST be orthogonal.

not too scale

(a) Compute the gradient $\nabla f(-3, 2)$, and plot $\nabla f(-3, 2)$ with its tail at the point $(-3, 2)$.

$f_x(x,y) = -2x$, $f_y(x,y) = -8y$ so $\nabla f(-3,2) = \langle -2(-3), -8(2) \rangle$

$= \boxed{\langle 6, -16 \rangle}$

(b) Focus now on the contour $f(x, y) = 0$ which contains the point $(-3, 2)$.

i. Give a parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ of this curve. A complete answer gives both the vector function $\mathbf{r(t)}$ and the domain of values for t.

$f(x,y) = 0 \Rightarrow 25 = x^2 + 4y^2$ or $1 = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{(5/2)}\right)^2$

Thus $\boxed{\vec{r}(t) = \langle 5\cos t, \frac{5}{2}\sin t \rangle}$ To get the full ellipse, $0 \le t \le 2\pi$.

To trace it more than once $t \ge 0$.

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$\vec{r}(t) = \langle 5\cos t, \frac{5}{2}\sin t \rangle$

ii. Give the tangent vector $\mathbf{r}'(t)$ at the point $(-3, 2)$ shown as a black dot.

$\vec{r}\,'(t) = \langle -5\sin t, \frac{5}{2}\cos t \rangle$

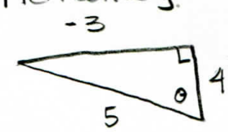
To find θ such that, $\vec{r}(\theta) = (-3, 2)$, Solve

$5\cos\theta = -3$ and $\frac{5}{2}\sin\theta = 2$

i.e. $\cos\theta = -3/5$ and $\sin\theta = \frac{4}{5}$.

Since I took $0 \le t \le 2\pi$, at $(-3,2)$, $\vec{r}\,'(t)$ should point (roughly) Southwest. It is drawn in purple on the contour plot.

Alternatively ...



You do not need the exact value of θ since

$$\vec{r}\,'(\theta) = \langle -5\sin\theta, \frac{5}{2}\cos\theta \rangle$$
$$= \langle -5\left(\frac{4}{5}\right), \frac{5}{2}\left(\frac{-3}{5}\right) \rangle$$
$$= \boxed{\langle -4, -3/2 \rangle}$$

(c) Without doing any work at all, find the value of the dot product of the tangent vector at $(-3, 2)$ and the gradient vector $\nabla f(-3, 2)$. Succinctly, explain your answer.

0 Since the gradient vector $\nabla f(-3,2)$ is orthogonal to the level curve $25 = x^2 + 4y^2$ containing $(-3, 2)$

(d) Now show all work justifying your previous answer.

Work: Compute $\nabla f(-3,2) \cdot \vec{r}\,'(\theta)$

$$= \langle 6, -16 \rangle \cdot \langle -4, -3/2 \rangle$$
$$= 6(-4) + (-16)(-3/2)$$
$$= -24 + 24$$
$$= \underline{\underline{0}}$$

as anticipated.

2. Consider the two surfaces (I) and (II) given below:

$$(I) \quad g(x,y) = 2x\,e^{2x-3y^2} \qquad\qquad (II) \quad z^2 - y\sin\left(\frac{\pi x}{12}\right) = 35$$

It is easy to check that the point $(3, \sqrt{2}, 6)$ lies on both surfaces.

(a) Consider the **two** tangent planes to these **two** surfaces at the point $(3, \sqrt{2}, 6)$. Explain clearly and precisely how you would find the normal vectors to the **two** tangent planes.

For (I), I find the normal vector by ....

Recognize that $g(x,y) = z = 2xe^{2x-3y^2}$ gives the surface $(x, y, g(x,y))$ parametrically. The equation of the tangent plane is given by $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$

OR $z = f(3,\sqrt{2}) + f_x(3,\sqrt{2})(x-3) + f_y(3,\sqrt{2})(y-\sqrt{2})$ or, equivalently,

$6 = f_x(3,\sqrt{2})(x-3) + f_y(3,\sqrt{2})(y-\sqrt{2}) - z$. Reading off $\vec{n}$ we find $\vec{n}$ is parallel to $\langle f_x(3,\sqrt{2}), f_y(3,\sqrt{2}), -1\rangle$.

For (II), I find the normal vector by ....

This surface is given IMPLICITLY. It can be viewed as a LEVEL SURFACE $G(x,y,z) = 35$ where $G(x,y,z) = z^2 - y\sin\left(\frac{\pi x}{12}\right)$

Thus, the normal direction $\vec{n}$ is given by any non-zero scalar multiple of $\nabla G(3,\sqrt{2},6)$.

(b) Find the equation of the tangent plane at $(3, \sqrt{2}, 6)$ for the surface given by (I).

$g(x,y) = 2xe^{2x-3y^2}$ so $g_x(x,y) = 2\left[xe^{2x-3y^2}(2) + (1)e^{2x-3y^2}\right]$

$= 2e^{2x-3y^2}(2x+1)$.

Also, $g_y(x,y) = 2xe^{2x-3y^2}(-6y) = -12xye^{2x-3y^2}$. Evaluating at $(3,\sqrt{2})$,

we find $g_x(3,\sqrt{2}) = 2e^0(2(3)+1) = 14$ and $g_y(3,\sqrt{2}) = -12(3)(\sqrt{2})e^0 = -36\sqrt{2}$.

Of course, $g(3,\sqrt{2}) = 6$. Thus,

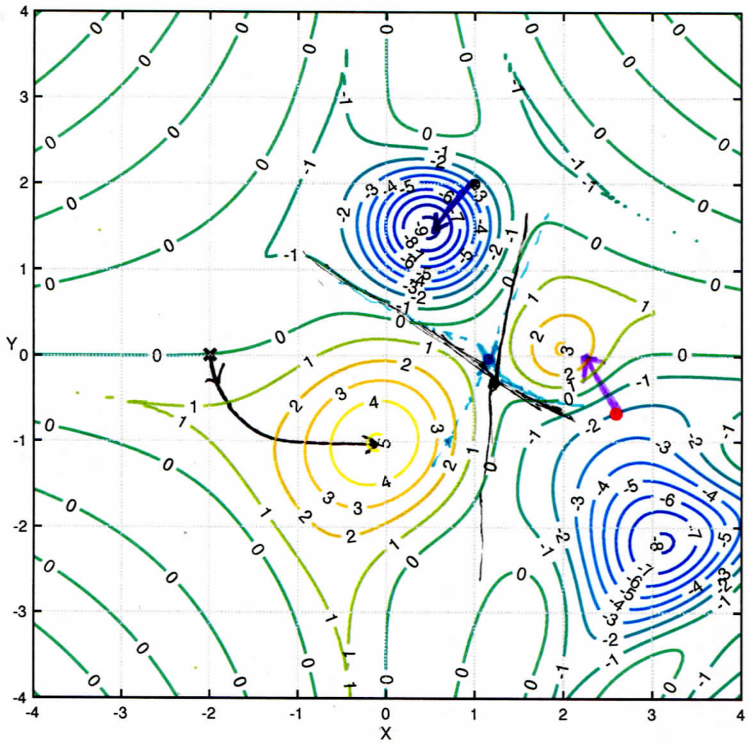
$z = g(3,\sqrt{2}) + g_x(3,\sqrt{2})(x-3) + g_y(3,\sqrt{2})(y-\sqrt{2})$

$\Rightarrow z = 6 + 14(x-3) + (-36\sqrt{2})(y-\sqrt{2})$

$\Rightarrow z = 6 + 14x - 42 - 36\sqrt{2}y + 72$

$\Rightarrow \boxed{14x - 36\sqrt{2}y - z = -36}$

3. Consider the contour plot for the smooth function $z = f(x, y)$ displayed below.



purple vector

(a) At the red point $(2.6, -0.7)$ shown, draw a vector pointing in the direction of $\nabla f(2.6, -0.7)$.

ORTHOGONAL TO LEVEL CURVE, points in direction of maximal increase in $f(x,y)$.

(b) Consider the black point $(1, 2)$ shown in the contour plot.
Estimate $f_{\vec{u}}(1, 2)$ where $\vec{u}$ points in the direction of $\vec{v} = \langle -1, -1 \rangle$.

$\vec{u}$ a unit vector $\vec{u} = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$. See figure.

$$f_{\vec{u}}(1,2) \approx \frac{\Delta f}{|\vec{u}|} = \frac{-9-{}^{-}3}{1} = -6$$

↑ variation possible of course.

Should be close to this however.

(c) The function $f(x, y)$ has (at least) one saddle point at (a, b). Give the coordinates (a, b) for this saddle point and then **justify** why this is a saddle point for $f(x, y)$.

X = (1, 0) or really (1.1, 0). Informal justification: two local maxes in yellow, two local mins in blue shown with (1.1, 0) is the middle. (See dotted lines.) or black lines. . . .

(1.1, −.2) also good. This is an estimate.

Formal justification: $f(x,y)$ decreases as you move towards the local mins, increases as you move towards the local max's.

(d) Suppose a negatively charged particle is placed at the black X at $(-2, 0)$, and that $f(x, y)$ gives the charge of a plate in coulombs. Sketch the path the negatively charged particle on the plate.

"follow the gradient" The path should be orthogonal to level curves.