MATH 253 Name: ______________________________

Final exam April 27, 2021

Instructions: 110 points total plus extra credit. You get your grade out of 100. Use only your brain and writing implement. You have 120 minutes to complete this exam. Good luck and happy summer!

1. (9 pts. – 3 pts. each) Consider the three non-coplanar points $A(2,1,0)$, $B(0,3,1)$, and $C(2,2,-1)$ in $\mathbb{R}^3$.

   (a) Find the area of the parallelogram spanned by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

   (b) Find the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and determine if it is acute, right, or obtuse.

   (c) Find the equation of the plane that contains the three points A, B, and C.

2. (6 pts.) A particle moves in $\mathbb{R}^3$ on a trajectory given by

$$\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 3t^2\,\mathbf{j} + 2\cos t\,\mathbf{k} \quad \text{for } t \geq -2$$

   where x, y, and z are measured in meters and $t \geq -2$ is measured in seconds.

   (a) Give a formula for the **speed** of the particle at any time t. Include units in your answer.

   (b) At what time t is the speed of the particle at a minimum and what is the speed at that time?

3. (15 pts.) Consider the function $f(x,y) = (x-1)(e^{y+2}-1)$.

(a) (6 pts.) Find all critical points of $f(x,y)$. Then classify the critical points as local maxima, minima or saddle points, and give the value of $f(x,y)$ at these points.

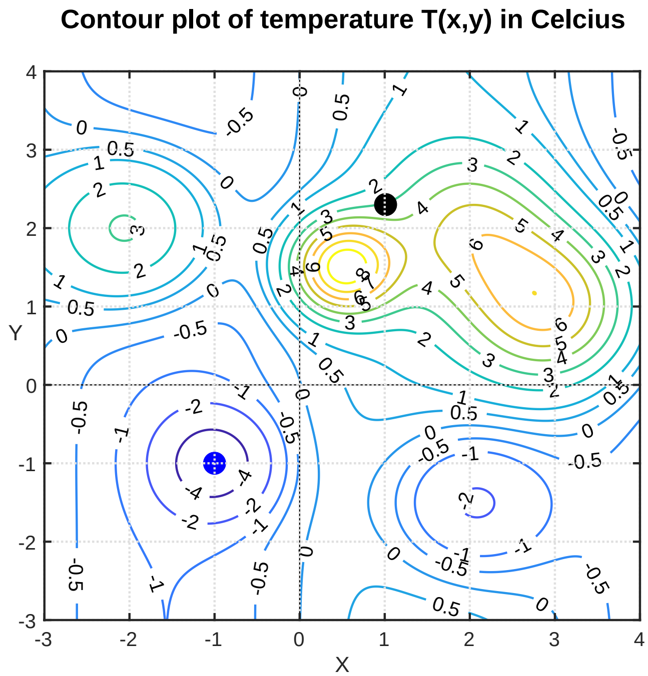
(b) (4 pts.) Find the directional derivative $D_{\mathbf{u}}f$ at the point $P(0, \ln 3 - 2)$ in the direction of $\mathbf{v} = \langle -3,\, 4\rangle$.

(c) (2 pts.) In what direction should you move from the point $P(0, \ln 3 - 2)$ to increase the value of $f(x,y)$ the most? the least? Explain briefly.

(d) (3 pts.) Find the equation of the tangent plane to $f(x,y)$ at the point

$$P\big(0,\, \ln 3 - 2,\, f(0, \ln 3 - 2)\big)$$

4. (6 pts.) Consider the contour plot of a function $T(x, y)$ that gives the temperature in Celcius of a particle in $\mathbb{R}^2$.



(a) (2 pts.) Give the coordinates (a, b) of a saddle point of $f(x, y)$ (or mark it in the figure). Explain why $f(x, y)$ has a saddle point at (a, b).

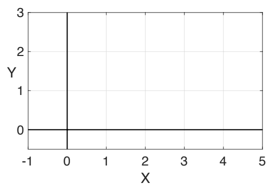
(b) Two points are marked on the contour plot in the colors black and blue.

i. (2 pts.) Suppose a heat-seeking particle is placed at the **black** point $(1, 2.3)$. Sketch on the contour plot the path of the particle as it moves to maxmize its heat. Briefly explain why you drew the trajectory the way you did.
Explanation:

ii. (2 pts.) At the **blue** point $(-1, 1)$, determine if the following is positive / negative / zero. Justify your answer.

$f_{xx}(-1, 1)$ is ________________ because ....

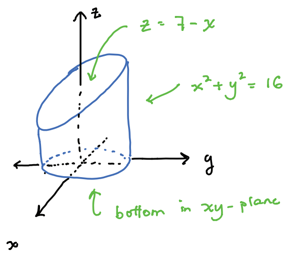
5. (8 pts.) Sketch the region of integration on the axes below. Then, by interchanging the order of integration, compute the following double integral.



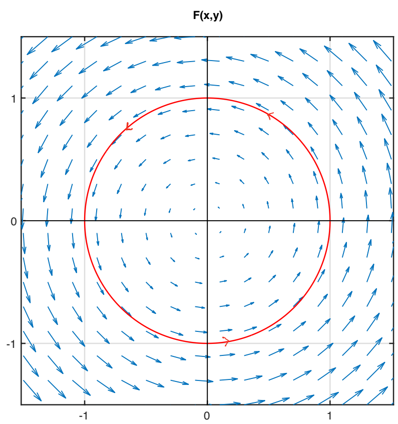
$$\int_0^4 \int_{\sqrt{x}}^2 \sqrt{y^3+1}\, dy\, dx$$

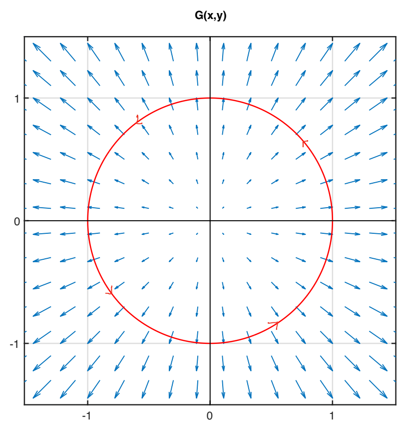
6. (8 pts.) Consider the 2-dimensional force field $\mathbf{F}(x,y) = \left\langle \cos x - x^2 y,\, \sin y + xy^2 \right\rangle$ Newtons and x and y are measured in meters. Suppose that a particles starts at the origin, moves in a straight line to the point $\left(\frac{-3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}\right)$, then **clockwise** part way around the circle to $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$, then on the straight line segment back to the origin. See Figure. Use Green's Theorem to compute the work done by the force field on this particle. Include units in your answer.

7. (5 pts.) Give a parameterization in cylindrical coordinates of the solid E bounded by the xy-plane, the cylinder $x^2 + y^2 = 16$, and the plane $z = 7 - x$. Be sure to give the range of values for your parameters.



8. (10 pts.) Below are pictured two vectors fields, $\mathbf{F}(x, y)$ and $\mathbf{G}(x, y)$, and the positively oriented unit circle C in red.





(a) (4 pts.) Consider the circulation $\left(\oint_C \mathbf{F} \cdot d\mathbf{r},\ \oint_C \mathbf{G} \cdot d\mathbf{r}\right)$, of these vector fields around C. Determine whether the values of these line integrals are (positive/negative/zero). **Justify** briefly your answers.

(b) (3 pts.) One of the two vector fields is conservative and the other is not. Which one must be the conservative one? Why?

(c) (3 pts.) Consider the point $(-1, -1)$ (not marked) on the plot of $\mathbf{G}$ on the right. Give an estimate for $\operatorname{curl} \mathbf{G}(-1, -1)$. Explain briefly.

9. (20 pts) Consider the sphere S of radius 2 and its interior the solid E, and the vector field

$$\mathbf{F}(x, y, z) = \langle -5x,\ -5y, -5z \rangle$$

defined on all of $\mathbb{R}^3$. In this problem you will be stepped through verifying the Divergence Theorem if you do all the parts including the extra credit.

(*N.B.:* Most parts of this problem are independent of one another. Make sure to try them all.)

(a) (5 pts.) By the Divergence Theorem the flux integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ has the same value as the triple integral

$$\iiint_E \operatorname{div} \mathbf{F} \, dV$$

Compute this triple integral.

[*Hint:* You can compute the value of this triple integral *very* easily, in about 30 seconds, if you think about it and include justification of your quick answer for full credit.]

(b) (2 pts.) Your answer from part (a) should not be zero. (If it was, do that part again.) Give a physical interpretation of the meaning of the sign of the flux integral.

(c) (3 pts.) Give a parameterization $\mathbf{r}$ of the sphere $S = \partial E$ of radius 2. A complete answer gives $\mathbf{r}$ as a vector function of two parameters *AND* gives ranges of values for those parameters.

Answer: ____

(d) (2 pts.) Give a **unit** vector $\mathbf{n}$ that is normal to the sphere S and points outward.

[*Hint:* this can be done without computation if you think about it correctly.]

Answer: $\mathbf{n} =$ ____________________

(e) (5 pts.) For the parameterization of the sphere S you gave in the part (c), compute the quantity $d\mathbf{S}$.

(f) (**Extra credit**) By the Divergence Theorem, the flux of $\mathbf{F}$ through the surface S was computed in part (a). For extra credit, on a scrap piece of paper, compute the flux integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

checking that your answer agrees with that of (a) of course.

(g) (3 pts.) Now consider the unit sphere U and the flux integral of $\mathbf{F}$ through U. Without computing anything, explain whether

$$\left|\iint_U \mathbf{F} \cdot d\mathbf{S}\right| \text{ is LARGER / SMALLER / or EQUAL in value to } \left|\iint_S \mathbf{F} \cdot d\mathbf{S}\right|.$$

A correct answer must include a brief, but filled with specifics, justification of your answer. [Make note of the absolute value signs.]

10. (15 pts) A smooth vector field $\mathbf{F} = \left\langle e^{-xy} - xye^{-xy},\ -x^2e^{-xy} + \cos(y) \right\rangle$ is defined on the simply connected region $\mathbb{R}^2$.

(a) (3 pts.) By computing appropriately chosen partial derivatives of the component functions P, Q where $\mathbf{F} = \langle P, Q\rangle$, prove that $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^2$.

(b) (6 pts.) Find a potential function $f(x, y)$ for $\mathbf{F}$. [*Hint:* By making the right choices, you can minimize your work.]

(c) (3 pts.) Find the value of the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the line segment joining the point $P(0, \frac{\pi}{2})$ to the point $Q(1, \frac{3\pi}{2})$. (C is oriented from P to Q.)

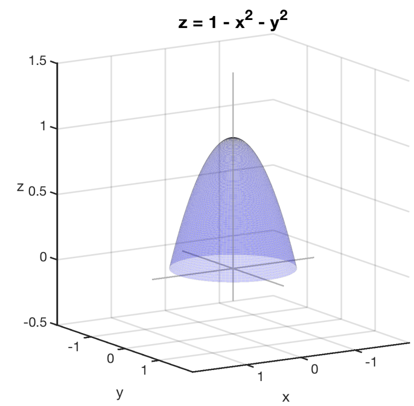
(d) (3 pts.) Now find the value of the line integral $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$ where C' is the semi-circular arc joining Q to P. Justify your answer.

11. (8 pts.) Consider the vector field $\mathbf{F}(x, y, z) = \langle x^2 \sin(z),\, y^2,\, xy \rangle$. By Stoke's Theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane. (Do not forget to check that the boundary is positively oriented which will be given by the 'natural' parameterization of C.)

(a) (3 pts.) Compute $\operatorname{curl} \mathbf{F}$.



(b) (5 pts.) Use Stoke's Theorem to compute a line integral that gives the value of

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

(c) **(Extra credit)**
Compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

by evaluating a surface integral. [*Hint:* By Stoke's Theorem, you can replace the paraboloid S by a surface that might ease your computation. Justify this if you follow this path.]