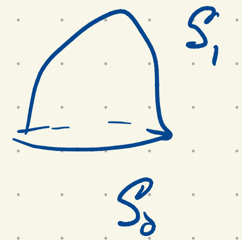
e.g. $\vec{X} = x\hat{i} + y\hat{j} + z\hat{k}$

compute flux of $\vec{X}$ out of



$x^2+y^2+z^2=4$

normal on S_1: $\frac{1}{2}\langle x, y, z\rangle$

$$\vec{X}\cdot\vec{n} = \frac{x^2+y^2+z^2}{2} = 2$$

On S_0, $z=0$, $\vec{n} = -k$ $\quad \vec{X}\cdot\vec{n} = 0$.

$$\iint_S \vec{X}\cdot\vec{n} = \iint_{S_1} 2\, dS = 2\cdot\frac{1}{2}\, 4\pi\cdot 2^2 = 16\pi$$

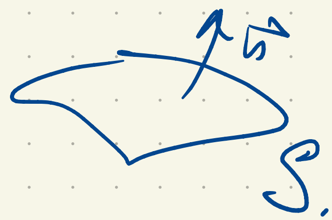
$\operatorname{div} X = 3 \qquad V = \frac{1}{2}\,\frac{4}{3}\pi\cdot 2^3 = \frac{16\pi}{3}$

$$\iiint_E \operatorname{div} X\, dV = \frac{16\pi}{3}\cdot 3 = 16\pi \;\checkmark$$

Maxwell's Equations relate two vector fields $\vec{E}, \vec{B}$ and a charge density ρ and a current density $\vec{J}$.

(Think of $\vec{J}$ as like a mass flux but it is charge)

$$\iint_S \vec{J}\cdot\vec{n}\, dS = \text{flux of charge } (C/s) \text{ through}$$



$$\iiint_{\mathcal{E}} \rho\, dV \quad \text{total charge contained in } \mathcal{E}.$$

$$\frac{d}{dt}\iiint_{\mathcal{E}} \rho\, dV = -\iint_S \vec{J}\cdot\vec{n}\, dS$$

$$= -\iiint_{\mathcal{E}} \operatorname{div} J\, dV$$

$$\iiint_{\mathcal{E}} (\rho_t + \operatorname{div} J)\, dV = 0 \Rightarrow \overbrace{\rho_t + \operatorname{div} J = 0}^{\text{continuity equation.}}$$

$\vec{E}$, electric field

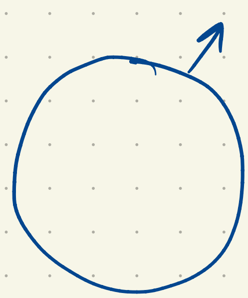
$[E] = N/C$

charge q experiences a force $q\vec{E}$.

$$\int_S \vec{E}\cdot d\vec{r}$$: work done transporting unit charge (1 C) over the curve.

To transport any other charge, multiply by q.

Gauss' Law



$$\varepsilon_0 \iint_S \vec{E}\cdot\vec{n}\, dS = \iiint_{\mathcal{E}} \rho \, dV$$

$[\varepsilon_0] = \dfrac{C^2}{m^2 N}$.

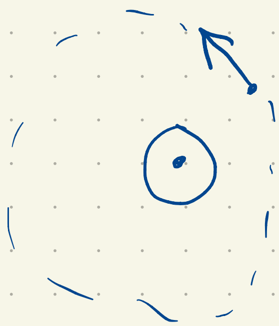
(charge generates electric flux over a closed surface)

$$\iiint \operatorname{div} E \, dV = \iiint \rho \, dV \Rightarrow \boxed{\operatorname{div} E = \rho.}$$

Magnetic field $\vec{B}$.

Force on a moving particle with charge q: $q\vec{v}\times\vec{B}$

velocity $\vec{v}$



$$[B] = \frac{N}{C}\frac{s}{m}$$

Never any magnetic flux:

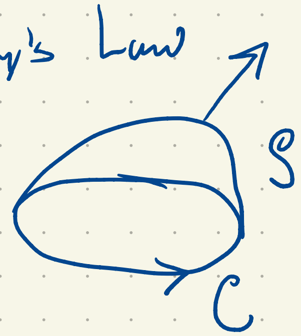
$$\iint_S \vec{B}\cdot\vec{n}\,dS = 0$$

$$\boxed{\operatorname{div}\vec{B} = 0}$$

(Gauss' law for mag field)

No work $q\vec{v}\times\vec{B}\cdot\vec{v} = 0$

Faraday's Law

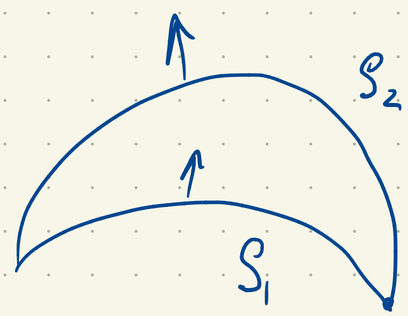


$$\int_C E\cdot d\vec{r} = -\frac{d}{dt}\iint_S \vec{B}\cdot\vec{n}\,dA$$

What is $\int_C E \cdot d\vec{r}$? work done by moving charge around loop.

If loop is a wire, a positive charge will traverse C in its orientation

You can induce a current in a wire by changing the magnetic field inside the loop,



$$\int_{S_2} B \cdot \vec{n}\, dS = \int_{S_1} B \cdot \vec{n}\, dS \ ?$$

$$\int_S B \cdot \vec{n}\, dS = 0 \ \checkmark$$

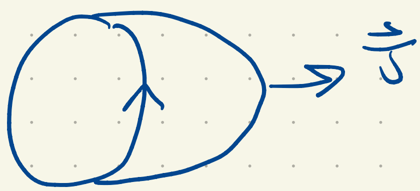
$$\left(\operatorname{div} \vec{B} = 0 \ \checkmark \right)$$

$$\int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot \vec{n}\, dS$$

$$\iint_S (\nabla \times E) \cdot \vec{n}\, dS = -\frac{d}{dt} \iint_S B \cdot \vec{n}\, dS$$

$$\Rightarrow -\frac{dB}{dt} = \nabla \times E \qquad \text{(Faraday's Law)}$$

Finally: current generates a magnetic field



$$\int_C B \cdot d\vec{r} = \mu_0 \iint_S \vec{J} \cdot \vec{n} \qquad [\mu_0] = \frac{N s^2}{C^2}$$

$$\iint_S \vec{\nabla} \times \vec{B} \cdot \vec{n} = \mu_0 \iint_S \vec{J} \cdot \vec{n}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

But not quite

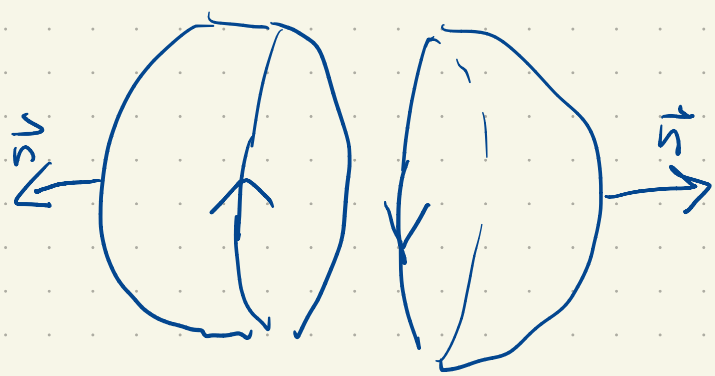
$$0 = \nabla \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$$

$$\rho_t + \mu_0 \nabla \cdot \vec{J} = 0 \Rightarrow \rho_t = 0.$$

Charge can't move?

There is a missing term

A changing electric field can also generate a magnetic field



$$\iint_S \vec{J} \cdot \vec{n} = 0 \quad ?$$

$$\hookrightarrow \frac{d}{dt} \iiint_{\mathcal{E}} \rho = -\frac{d}{dt} \iiint \varepsilon_0 \,\mathrm{div}\, E \, dV$$

$$= -\frac{d}{dt} \iint_S \varepsilon_0 E \vec{n} \, dS$$

$$\iint_S \left(\vec{J} + \frac{d}{dt}\varepsilon_0 E\right) \cdot \vec{n}\, dS = 0$$

$$\int_C \vec{B} \cdot d\vec{r} = \mu_0 \iint_S \left(\vec{J} + \frac{d}{dt}\varepsilon_0 E\right) \cdot \vec{n}$$

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot \vec{n} = \mu_0 \iint_S (J + \varepsilon_0 E_t) \cdot \vec{n}$$

$$\mu_0 \varepsilon_0 \frac{d}{dt}\vec{E} + \mu_0 \vec{J} = \vec{\nabla} \times \vec{B}$$

$$\downarrow$$

$$\frac{s^2}{m^2} \rightarrow \frac{1}{c^2}$$

$$\frac{1}{c^2}\frac{d}{dt}\vec{E} + \mu_0 \vec{J} = \nabla \times B$$

$$\frac{1}{\varepsilon_0}\frac{d}{dt} \operatorname{div} E + \operatorname{div} J = 0$$

$$\rho_t + \operatorname{div} J = 0 \quad \checkmark$$

☺

$$\varepsilon_0 \operatorname{div} E = \rho$$

$$\operatorname{div} B = 0$$

$$\nabla \times E = \frac{dB}{dt}$$

$$\nabla \times B = \mu_0 \vec{J} + \frac{1}{c^2}\frac{dE}{dt}$$

$$\boxed{\rho_t + \operatorname{div} J = 0}$$