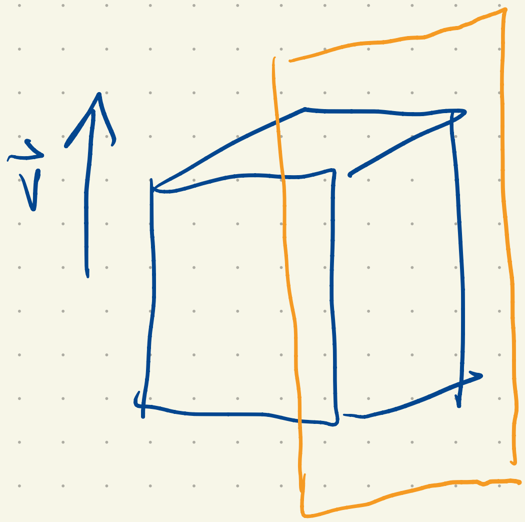
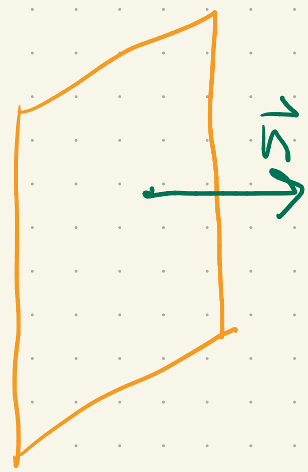
What if $\vec{v}$ is parallel to surface?



No fluid passes through

Zero flux.

What if $\vec{v}$ is neither perpendicular, nor parallel?



unit normal to surface.

$$\vec{v} = \vec{w} + c\vec{n} \qquad \vec{w}\cdot\vec{n} = 0$$

$$\vec{v}\cdot\vec{n} = \vec{w}\cdot\vec{n} + c\,\vec{n}\cdot\vec{n}$$

$$= 0 + c\cdot 1 = c$$

$$\vec{v} = \underset{\uparrow}{\vec{w}} + \underbrace{(\vec{v}\cdot\vec{n})\vec{n}}_{\text{source of all flux}}$$

parallel, no flux

$$\left| (\vec{v} \cdot \vec{n}) \vec{n} \right| = \left| \vec{v} \cdot \vec{n} \right|$$

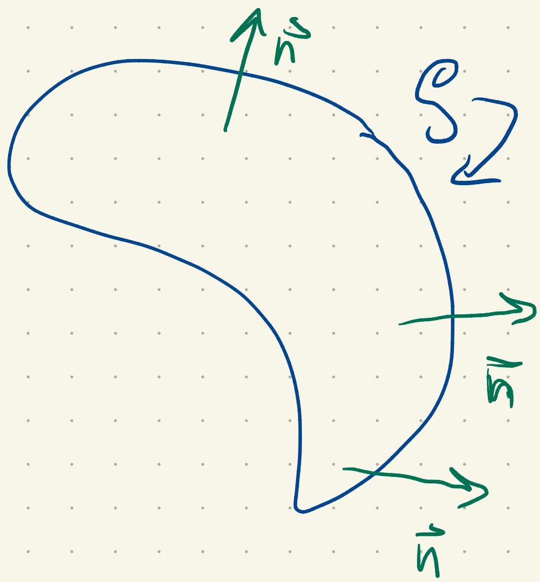
Mass flux: $\rho \left| \vec{v} \cdot \vec{n} \right| \Delta x \Delta y$

We'll drop the absolute values.

positive flux crosses surface in one direction.

negative flux crosses surface in opposite directions.

$\rho \vec{v} \cdot \vec{n} \Delta x \Delta y \rightarrow$ rate at which mass crosses a small region of surface with area $\Delta x \Delta y$.



$\rho \vec{v}$ not constant

surface not a plane

mass flux:

$$\boxed{\iint_S \rho \vec{v} \cdot \vec{n} \, dS}$$

(mass/time)

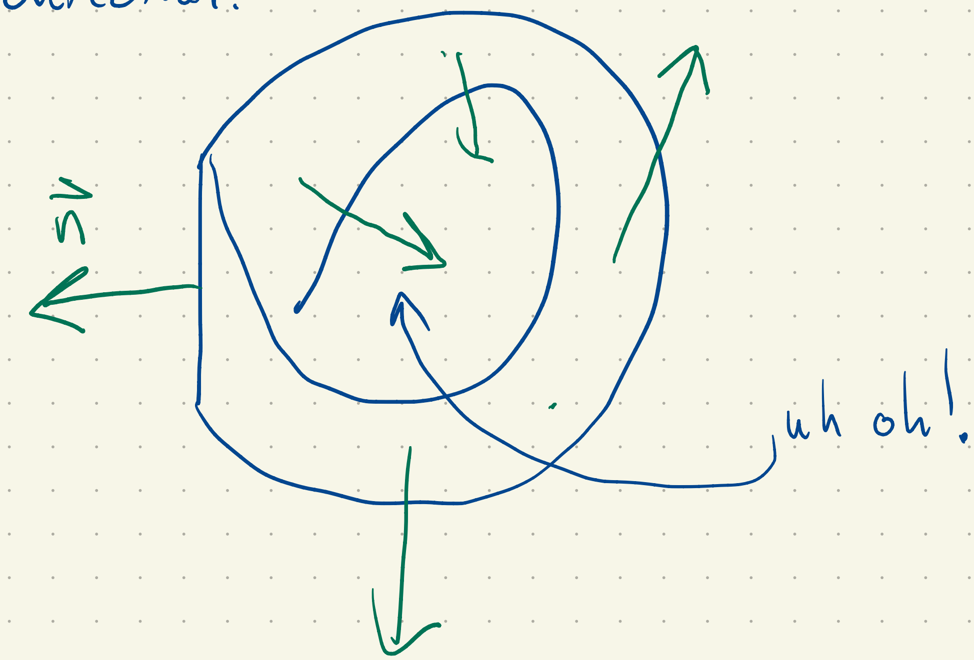
If $\vec{X}$ is any vector

$$\iint_S \vec{X} \cdot \vec{n} \, dS$$

flux of X through S

But: this involves a choice. Flux in which direction?

Not every surface lets you pick a direction.



We only compute flux integrals over surfaces where we can pick a "side" or a unit normal everywhere. Such surfaces are called <u>orientable</u>.

Interpretation: $\rho \vec{v}$ ← velocity ⟶ mass flux

~~mass~~ mass density

T : temperature field

∇T (gradient of temp)

$k \rightarrow$ thermal conductivity

$\frac{W}{mK}$

$$\underbrace{\iint_S -\vec{\nabla} T \cdot \vec{n} \, dS}$$:

$\hookrightarrow$ flow of energy across S in $\vec{n}$ direction (J/s, i.e. watts) due to heat transfer.

Electric field $\vec{E}$

$$\iint_S \vec{E} \cdot \vec{n} \, dS$$ electric flux across S

Gauss' law

$\vec{n}$

encloses a region

permittivity

$$\iint_S E \cdot \vec{n} \, dS = \text{enclosed electric charge} / \epsilon_0$$

Note: $$(\vec{X}\cdot\vec{n})\,dS = \vec{X}\cdot\frac{(\vec{r}_u\times\vec{r}_v)}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,du\cdot dv$$

$$= \vec{X}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$$\iint_D \left(\vec{X}(\vec{r}(u,v))\cdot \vec{r}_u\times\vec{r}_v\right)du\,dv$$

$$\vec{X} = z\hat{\imath} + y\hat{\jmath} + x\hat{k}$$

$$\vec{r}(u,v) = \langle C_u C_v,\ S_u C_v,\ S_v\rangle$$

$$\vec{r}_u = \langle -S_u C_v,\ C_u C_v,\ 0\rangle$$

$$\vec{r}_v = \langle -C_u S_v,\ -S_u S_v,\ C_v\rangle$$

$$\vec{r}_u\times\vec{r}_v = \langle C_u C_v^2,\ S_u C_v^2,\ C_v S_v\rangle$$

$$(\vec{X}\cdot \vec{r}_u\times\vec{r}_v) = C_u S_v C_v^2 + S_u^2 C_v^3 + C_u C_v^2 S_v$$

$$= 2C_u S_v C_v^2 + S_u^2 C_v^3$$

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi}^{\pi} \left[2\cos(u)\sin(v)\cos(v)^3 + S_u^2 C_v^3 \right] du\, dv$$

$$\pi \int_{-\frac{\pi}{2}}^{\pi/2} \cos^3 v\, dv = \pi \int_{-\frac{\pi}{2}}^{\pi/2} \left[1 - \sin^2 v\right] \cos(v)\, dv$$

$$= \pi \left[\sin(v) - \frac{\sin^3(v)}{3} \right]_{-\pi/2}^{\pi/2}$$

$$= \pi \left[2 - \frac{2}{3} \right]$$

$$= \frac{4\pi}{3}$$

$z = 1 - x^2 - y^2$

$z \geq 0$

$$X = \underbrace{\left(x\hat{i} + y\hat{j} + 2z\hat{k} \right)}_{\text{velocity}} \rho$$

const

$\vec{r}(u,v) = \langle u, v, 1-u^2-v^2 \rangle$

$\vec{r}_u = \langle 1, 0, f_u \rangle$

$\vec{r}_v = \langle 0, 1, f_v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1 \rangle$