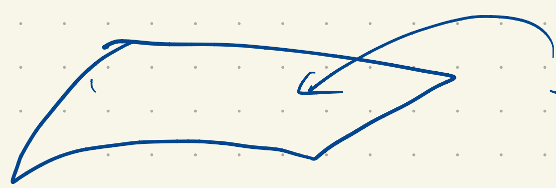
Surface integrals

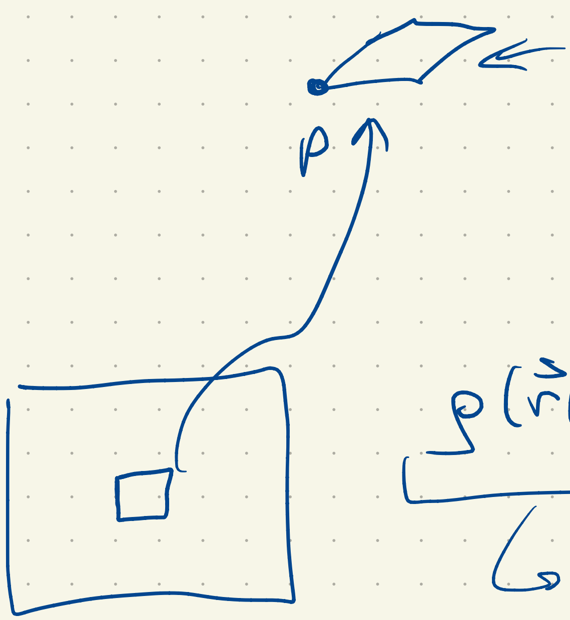
two kinds

a) don't care about orientations



$\rho(x,y,z)$ mass density

(mass per area



mass $\approx \rho(P)\,\Delta S$

↳ area.

$$\underbrace{\rho(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv}_{\text{we integrate this.}}$$

In effect: $dS \to |\vec{r}_u \times \vec{r}_v|\,du\,dv$

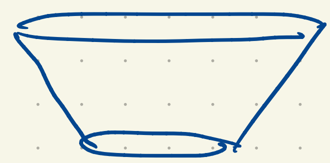
$$\rho(x,y,z) \to \rho(\vec{r}(u,v))$$

And bounds of integration need to describe domain in terms of u, v.

Resulting object $\iint_{S} g(x,y,z)\,dS$

e.g. $\iint_{\Sigma} z^2\,dS$

Σ



$z=\sqrt{x^2+y^2}$

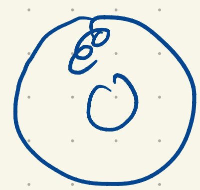
$1 \le z \le 2$

$\vec{r}(u,v) = \langle u, v, \sqrt{u^2+v^2}\rangle$

$dS = \sqrt{1+f_u^2+f_v^2}\,du\,dv$ $\qquad f(u,v)=\sqrt{u^2+v^2}$

$f_u = \frac{u}{\sqrt{u^2+v^2}}$ $\qquad f_v = \frac{v}{\sqrt{u^2+v^2}}$ $\qquad f_u^2+f_v^2 = \frac{u^2+v^2}{u^2+v^2} = 1$

$dS = \sqrt{2}\,du\,dv$



$$\iint_{\mathcal{E}} (u^2+v^2)\underset{\sqrt{2}}{\uparrow} dA = \sqrt{2}\int_0^{2\pi}\int_1^2 r^2\, r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{r^4}{4}\Big|_1^2 d\theta = 2\pi\left(4-\frac{1}{4}\right)\sqrt{2}$$

$$= 2\pi \left(\frac{15}{4}\right)\sqrt{2}$$

$$= \frac{15\sqrt{2}}{2}\pi$$

unit sphere

$$\iint_S x^2 \, dS$$

last class

$$dS = \cos v \, du \, dv$$

$$\vec{r}(u,v) = \langle \cos u \cos v, \sin u \cos v, \sin v \rangle$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\pi} \cos^2 u \cos^2 v \cos v \, du \, dv$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \cos^3 v \; \frac{1}{2} \, dv \cdot 2\pi$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} (1 - \sin^2 v) \cos(v) \, dv \cdot \pi$$

$$= \left(\sin(v) \Big|_{-\frac{\pi}{2}}^{\pi/2} - \frac{\sin^3(v)}{3} \Big|_{-\pi/2}^{\pi/2} \right) \cdot \pi$$

$$= \left(2 - \frac{2}{3}\right) \cdot \pi = \frac{4\pi}{3}$$

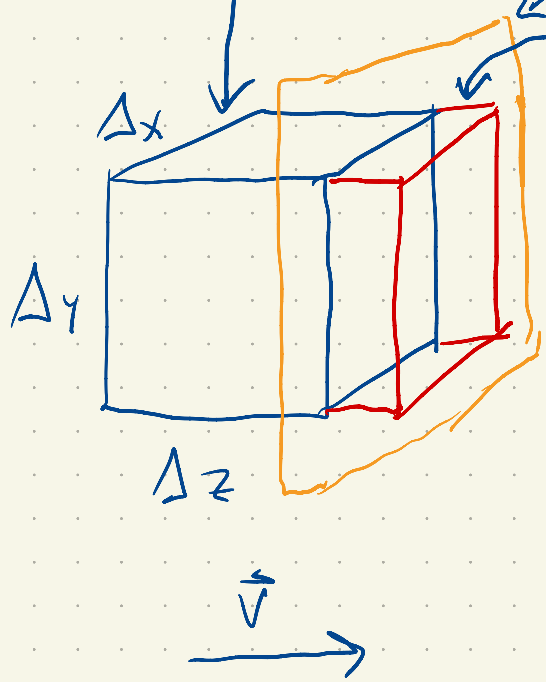
Flux

total mass: $\rho \Delta x \Delta y \Delta z$

surface fluid passes thru

mass in here:

$\rho \Delta x \Delta y |\vec{v}| \Delta t$

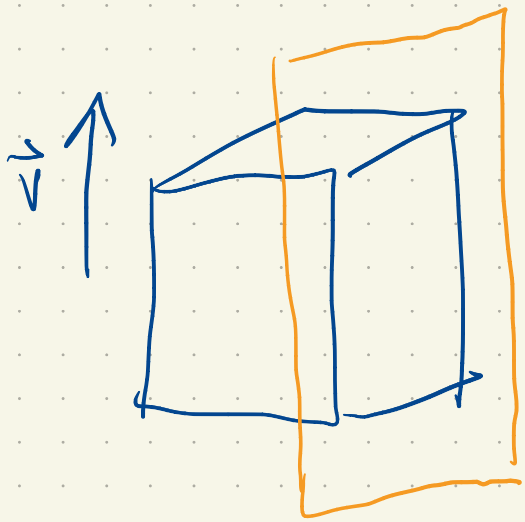


In time Δt, a total mass of $\rho \Delta x \Delta y |\vec{v}| \Delta t$ has passed through the plane.

Mass flux: rate at which mass passes through:

$$\rho |\vec{v}| \Delta x \Delta y$$ (units of mass per time).

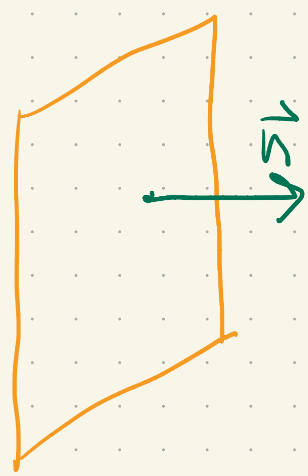
What if $\vec{v}$ is parallel to surface?



No fluid passes through

Zero flux.

What if $\vec{v}$ is neither perpendicular, nor parallel?



unit normal to surface.

$$\vec{v} = \vec{w} + c\vec{n} \qquad \vec{w}\cdot\vec{n} = 0$$

$$\vec{v}\cdot\vec{n} = \vec{w}\cdot\vec{n} + c\,\vec{n}\cdot\vec{n}$$

$$= 0 + c\cdot\underline{1} = c$$

$$\underbrace{\vec{w}}_{\text{parallel, no flux}} + \underbrace{(\vec{v}\cdot\vec{n})\,\vec{n}}_{\text{source of all flux}}$$

$$\vec{v} = \vec{w} + (\vec{v}\cdot\vec{n})\,\vec{n}$$

parallel, no flux

source of all flux

$$\left|(\vec{v}\cdot\vec{n})\,\vec{n}\right| = |\vec{v}\cdot\vec{n}|$$

Mass flux: $\rho\,|\vec{v}\cdot\vec{n}|\,\Delta x\,\Delta y$

We'll drop the absolute values.

positive flux crosses surface in one direction.

negative flux crosses surface in opposite direction.

$\rho\,\vec{v}\cdot\vec{n}\,\Delta x\,\Delta y$ → rate at which mass crosses a small region of surface with area $\Delta x\,\Delta y$.