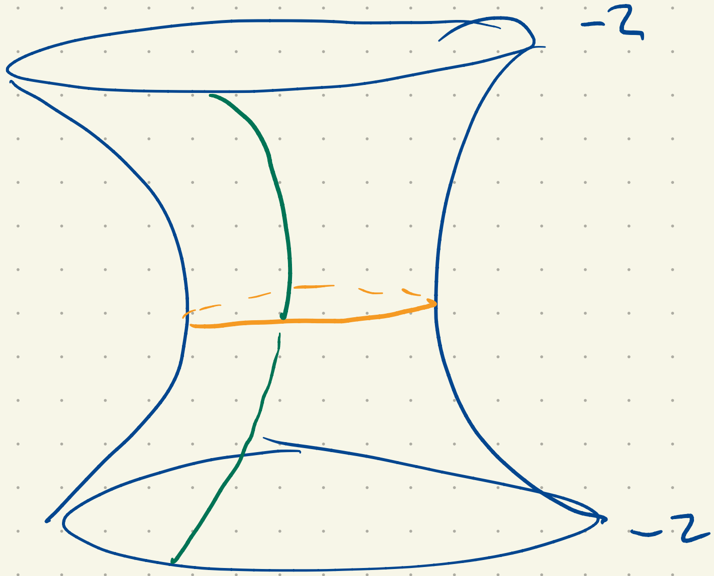
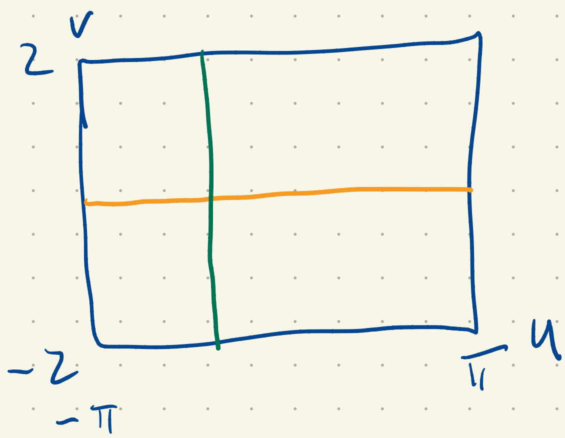


$$x^2+y^2 = z^2+1$$

$$r = \sqrt{z^2+1}$$



$$\left\langle \sqrt{v^2+1}\cos u,\ \sqrt{v^2+1}\sin u,\ v \right\rangle$$

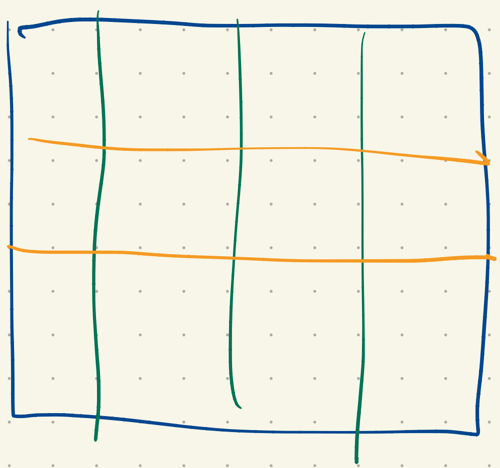
$$\uparrow$$

$$\vec{r}(u,v)$$

$$x^2+y^2 = v^2+1 = z^2+1 \checkmark$$

Less interesting but important

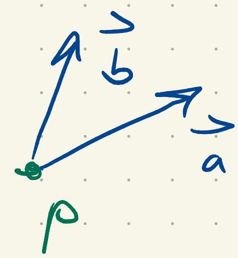
$$\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$$



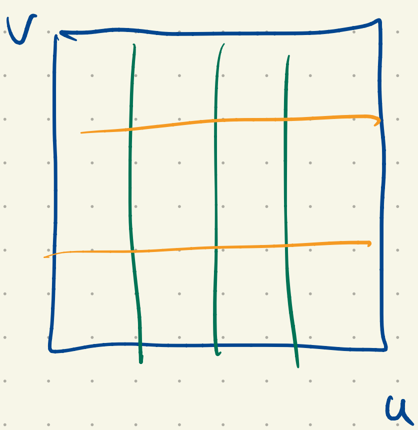
We can parametrize the graph of $f(x,y)$ with

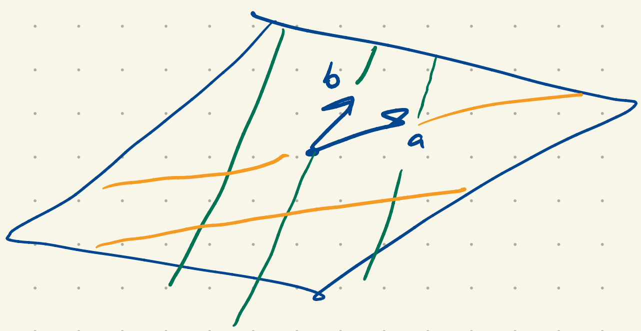
$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

in general

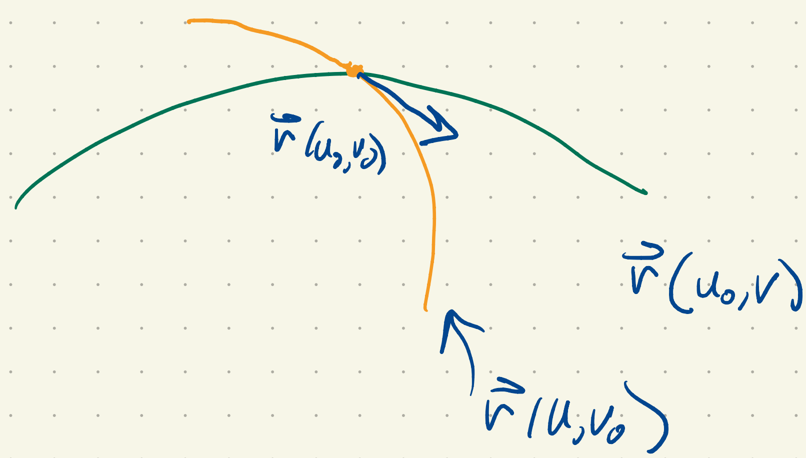


$$\vec{r}(u,v) = \vec{p} + u\vec{a} + v\vec{b}$$





A handy construction: normal vector

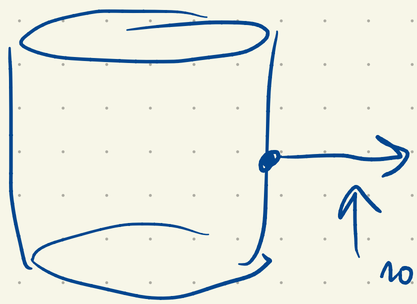


$\frac{\partial \vec{r}}{\partial u}(u_0,v_0)$, $\frac{\partial \vec{r}}{\partial v}(u_0,v_0)$ are tangent to the surface

$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is normal to the surface

It need not be a unit vector.

e.g. $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$



normal at $\langle 1,0,0 \rangle$ should point along x axis

$(u=0, v=0)$

$$\frac{\partial \vec{r}}{\partial u} = \langle -\sin(u), \cos(u), 0 \rangle$$

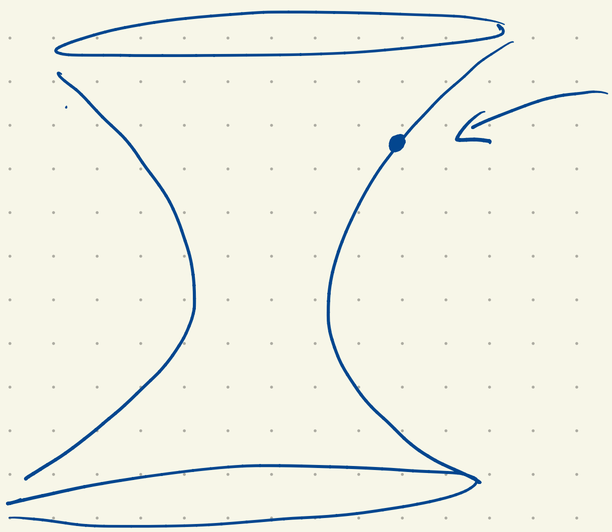
$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 0, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle \cos u, \sin u, 0 \rangle$$

and at $u=0, v=0$ this is $\langle 1, 0, 0 \rangle$ as advertized!

If you know the normal, you can also find the formula of the tangent plane.

e.g.



$\vec{r}(u,v) = \langle \sqrt{1+v^2}\cos u, \sqrt{1+v^2}\sin u, v \rangle$

$P = \langle \sqrt{10}, 0, 3 \rangle$

$u = 0,\ v = 3$

$x^2 + y^2 = z^2 + 1$ ✓

$$\frac{\partial \vec{r}}{\partial u} = \langle \sqrt{1+v^2}(-\sin u), \sqrt{1+v^2}\cos u, 0 \rangle$$

at P, $\frac{\partial \vec{r}}{\partial u} = \langle 0, \sqrt{10}, 0 \rangle$

$$\frac{\partial \vec{r}}{\partial v} = \langle \frac{v}{\sqrt{1+v^2}}\cos u, \frac{v}{\sqrt{1+v^2}}\sin u, 1 \rangle$$

$$= \langle \frac{3}{\sqrt{10}}, 0, 1 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \sqrt{10} & 0 \\ \frac{3}{\sqrt{10}} & 0 & 1 \end{vmatrix} = \hat{i}(\sqrt{10}) + \hat{j}\,0 + \hat{k}(-3)$$

$= \sqrt{10}\,\hat{\imath} - 3\hat{k}$

plane: $\sqrt{10}(x - x_0) + 0(y - y_0) - 3(z - z_0) = 0$

$\sqrt{10}(x - \sqrt{10}) - 3(z - 3) = 0$

Level set of $x^2 + y^2 - z^2$ $(= 3)$

$\nabla f = \langle 2x, 2y, -2z \rangle$

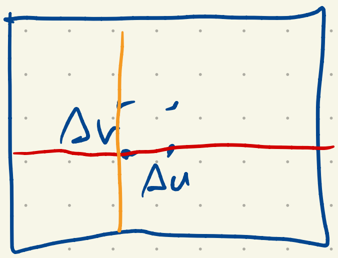
normal: $\langle x, y, -z \rangle$

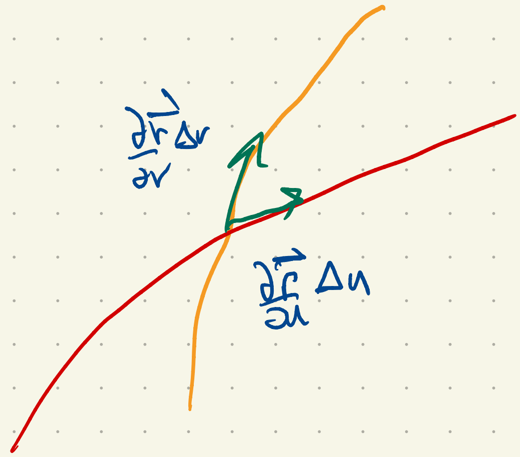
$x = \sqrt{10}$
$y = 0$
$z = 3$

$\langle \sqrt{10}, 0, -3 \rangle$

$\sqrt{10}(x - \sqrt{10}) - 3(z - 3) = 0$ ✓

Surface areas





$$\text{area} \approx \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \Delta u \Delta v$$

$$\iint_D \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| dA(u,v) = \text{surface area of } \vec{r}(D)$$

eg: Area of a sphere of radius R

$$\vec{r}(u,v) = \langle R\cos(u)\cos(v), R\sin(u)\cos(v), R\sin v \rangle$$

$$\frac{\partial \vec{r}}{\partial u} = \langle -R S_u C_v, R C_u C_v, 0 \rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \langle -R C_u S_v, -R S_u S_v, R C_v \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = R^2 \langle C_u C_v^2, S_u C_v^2, C_v S_v \rangle$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = R^2 \left[C_u^2 C_v^4 + S_u^2 C_v^4 + C_v^2 S_v^2 \right]^{1/2}$$

$$= R^2 \left[C_v^4 + C_v^2 S_v^2 \right]^{1/2}$$

$$= R^2 \left[C_v^2 \right]^{1/2}$$

$$= R^2 \cos(v)$$

$$\int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\pi/2} R^2 \cos(v)\, dv\, du = R^2 \int_{-\pi}^{\pi} \sin(v) \Big|_{-\frac{\pi}{2}}^{\pi/2} du$$

$$= 2\pi R^2 (1-(-1))$$

$$= 4\pi R^2 \quad (!)$$

$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

$$\frac{\partial \vec{r}}{\partial u} = \langle 1, 0, f_u \rangle$$

$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 1, f_v \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle -f_u, -f_v, 1 \rangle \longleftarrow \text{surface normal}$$