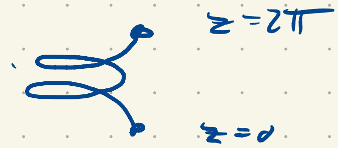
Section 16.6

Parametric Representation of Surfaces

Recall

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), t \rangle$$

$$0 \leq t \leq 2\pi$$



$$\vec{s}(t) = \langle \cos(s), \sin(s), s/2 \rangle$$

$$0 \leq s \leq 4\pi$$

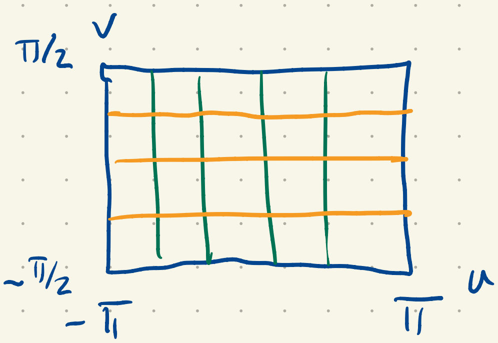
Different parameterizations of the same "curve".

They describe the points of the curve with different labels.

Now a 2-d version.

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

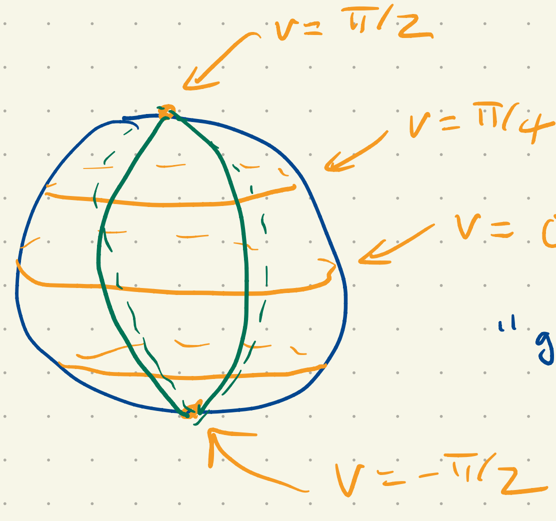
We're just going to do a bunch of examples.



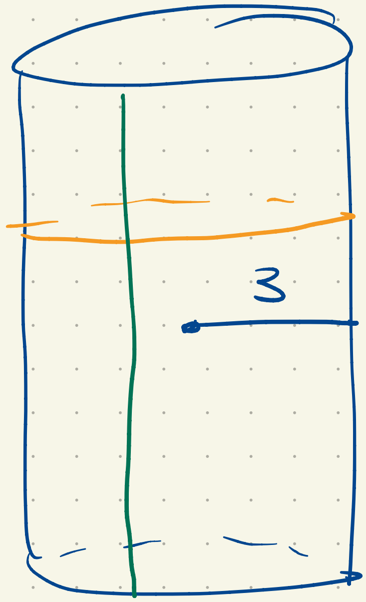
$$\vec{r}(u,v) = \langle \underbrace{\cos(u)\cos(v)}_{x(u,v)}, \underbrace{\sin(u)\cos(v)}_{y(u,v)}, \underbrace{\sin(v)}_{z(u,v)} \rangle$$

$x^2+y^2 = \cos^2 v$

$x^2+y^2+z^2 = \cos^2 v + \sin^2 v = 1$ ☺

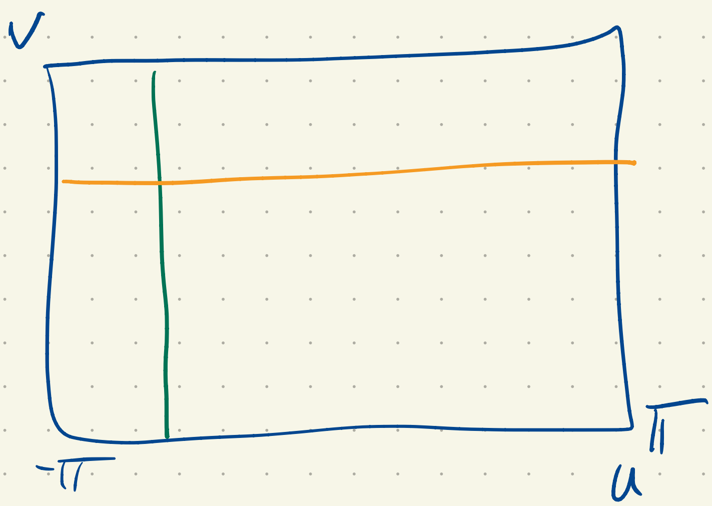


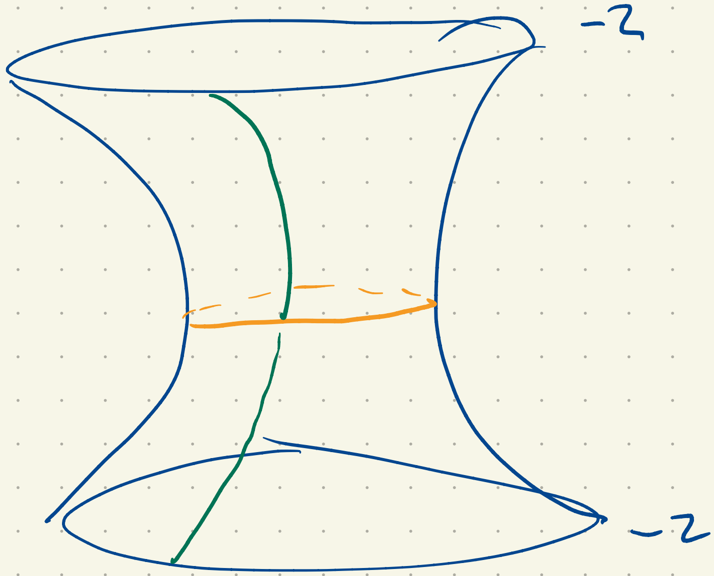
"grid lines"



$x^2+y^2=9$

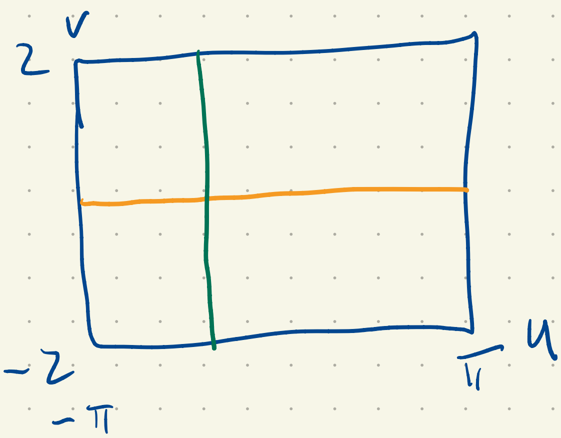
$\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle$





$$x^2+y^2 = z^2+1$$

$$r = \sqrt{z^2+1}$$



$$\langle \sqrt{v^2+1}\cos u, \sqrt{v^2+1}\sin u, v \rangle$$

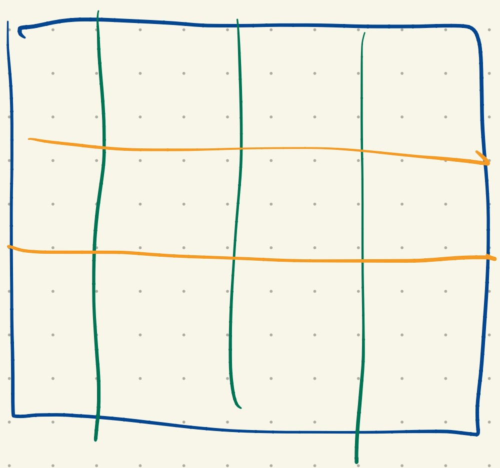
$$\uparrow$$

$$\vec{r}(u,v)$$

$$x^2+y^2 = v^2+1 = z^2+1 \checkmark$$

Less interesting but important

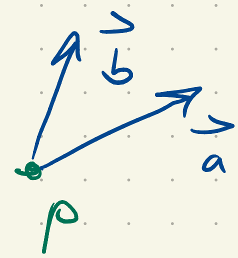
$\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$



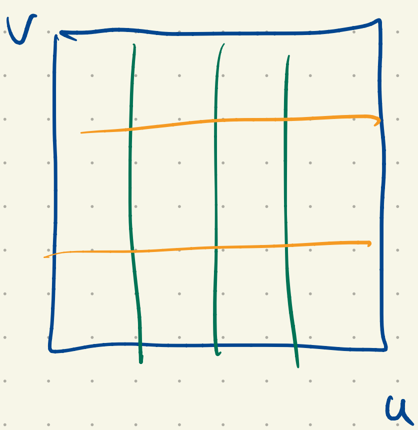
We can parametrize the graph of $f(x,y)$ with

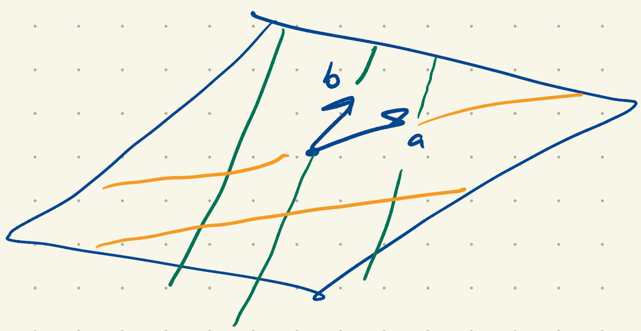
$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

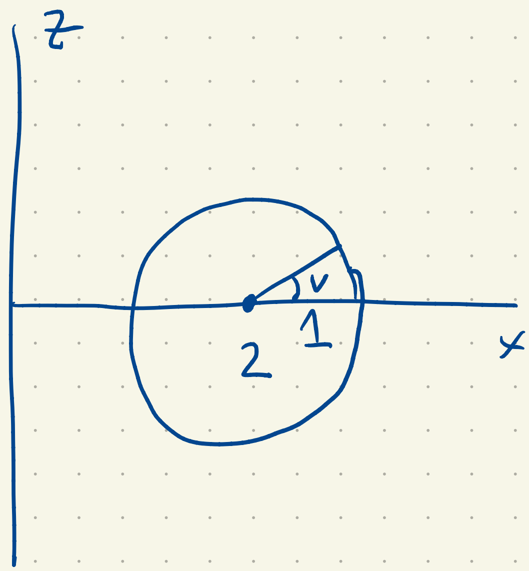
in general



$$\vec{r}(u,v) = \vec{p} + u\vec{a} + v\vec{b}$$



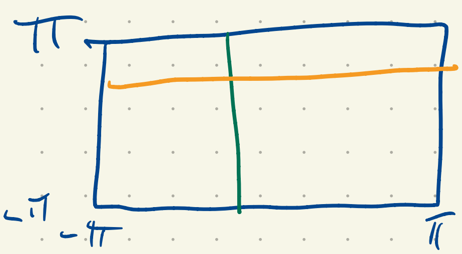




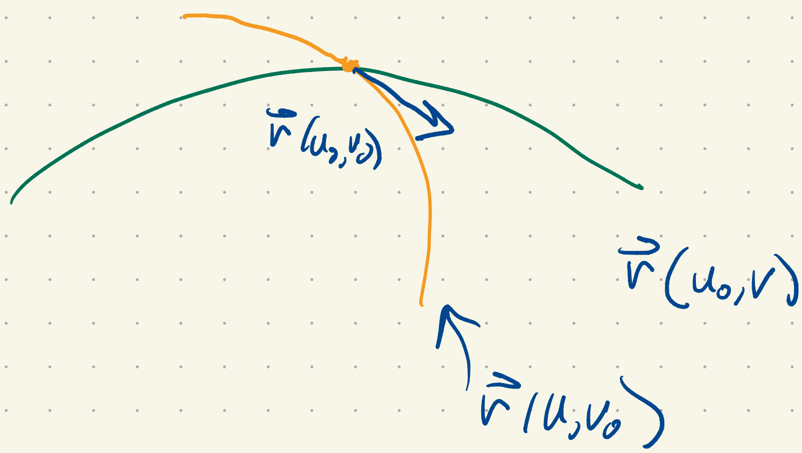
$$z = \sin v$$

$$x = 2 + \cos(v)$$

$$\vec{r}(u,v) = \left\langle (2+\cos(v))\cos u,\ (2+\cos(v))\sin u,\ \sin v \right\rangle$$



A handy construction: normal vector

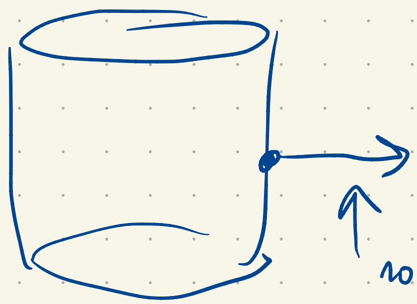


$\frac{\partial \vec{r}}{\partial u}(u_0,v_0)$, $\frac{\partial \vec{r}}{\partial v}(u_0,v_0)$ are tangent to the surface

$$\vec{N} = \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$ is normal to the surface

It need not be a unit vector.

e.g. $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$



normal at $\langle 1,0,0 \rangle$ should point along x axis

$(u=0, v=0)$

$$\frac{\partial \vec{r}}{\partial u} = \langle -\sin(u), \cos(u), 0 \rangle$$

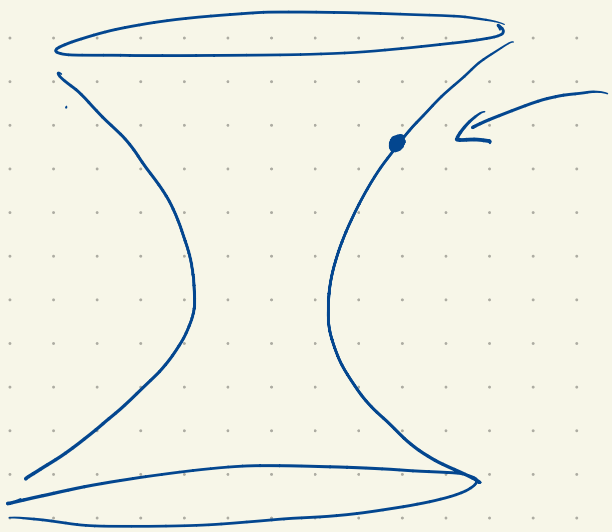
$$\frac{\partial \vec{r}}{\partial v} = \langle 0, 0, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \langle \cos u, \sin u, 0 \rangle$$

and at $u=0, v=0$ this is $\langle 1, 0, 0 \rangle$ as advertized!

If you know the normal, you can also find the formula of the tangent plane.

e.g.



$\vec{r}(u,v) = \langle \sqrt{1+v^2}\cos u, \sqrt{1+v^2}\sin u, v \rangle$

$P = \langle \sqrt{10}, 0, 3 \rangle$

$u = 0, v = 3$

$x^2 + y^2 = z^2 + 1$ ✓

$$\frac{\partial \vec{r}}{\partial u} = \langle \sqrt{1+v^2}(-\sin u), \sqrt{1+v^2}\cos u, 0 \rangle$$

at P, $\frac{\partial \vec{r}}{\partial u} = \langle 0, \sqrt{10}, 0 \rangle$

$$\frac{\partial \vec{r}}{\partial v} = \langle \frac{v}{\sqrt{1+v^2}}\cos u, \frac{v}{\sqrt{1+v^2}}\sin u, 1 \rangle$$

$$= \langle \frac{3}{\sqrt{10}}, 0, 1 \rangle$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \sqrt{10} & 0 \\ \frac{3}{\sqrt{10}} & 0 & 1 \end{vmatrix} = \hat{i}(\sqrt{10}) + \hat{j}\,0 + \hat{k}(-3)$$

$= \sqrt{10}\,\hat{i} - 3\hat{k}$

plane: $\sqrt{10}(x - x_0) + 0(y - y_0) - 3(z - z_0) = 0$

$\sqrt{10}(x - \sqrt{10}) - 3(z - 3) = 0$

Level set of $x^2 + y^2 - z^2$ $(= 3)$

$\nabla f = \langle 2x, 2y, -2z \rangle$

normal: $\langle x, y, -z \rangle$

$x = \sqrt{10}$
$y = 0$
$z = 3$

$\langle \sqrt{10}, 0, -3 \rangle$

$\sqrt{10}(x - \sqrt{10}) - 3(z - 3) = 0$ ✓
