Curl $\qquad X = \langle P, Q, R \rangle$

Last class $\operatorname{div} \vec{X} = \vec{\nabla} \cdot \vec{X} = \partial_x P + \partial_y Q + \partial_z R$

Now: second kind of derivative

$\operatorname{curl} \vec{X}$, vector.

a) How to compute

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (\partial_y R - \partial_z Q)\hat{i} - (\partial_x R - \partial_z P)\hat{j} + (\partial_x Q - \partial_y P)\hat{k}$$

Look at the $\hat{k}$ entry: $-\dfrac{\partial P}{\partial y} + \dfrac{\partial Q}{\partial x}$

we've seen this expression in Green's thm.

" $\vec{\nabla} \times \vec{X}$ "

e.g.

$$\vec{X} = xz\,\hat{i} + xy^2z\,\hat{j} - e^{2y}\,\hat{k}$$

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & xy^2z & -e^{2y} \end{matrix}$$

$$\text{curl}\,\vec{X} = \left(-2e^{2y} - xy^2\right)\hat{i} - \left(0 - x\right)\hat{j} + \left(y^2z - 0\right)\hat{k}$$

$$= \left(-2e^{2y} - xy^2\right)\hat{i} + x\,\hat{j} + y^2z\,\hat{k}$$

One application

$$\text{curl}\,\vec{\nabla} f = 0$$

$$\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{matrix}$$

$$\begin{aligned} \text{curl}\,\vec{\nabla} f = & \left(\partial_y \partial_z f - \partial_z \partial_y f\right)\hat{i} \\ & - \left(\partial_x \partial_z f - \partial_z \partial_x f\right)\hat{j} \\ & + \left(\partial_x \partial_y f - \partial_y \partial_x f\right)\hat{k} \end{aligned}$$

This is the right generalization of the necessary condition $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$.

In fact it is sufficient on simply connected domains. (Uses Stokes Thm coming soon).

E.g.

$$f = e^{xy} - y\sin(z)$$

$$f_x = ye^{xy}$$
$$f_y = xe^{xy} - \sin(z)$$
$$f_z = y\cos(z)$$

$$\vec{X} = \langle ye^{xy}, xe^{yx} - \sin(z), y\cos(z) \rangle$$

$$\partial_y \uparrow - \partial_x \uparrow = 0 \checkmark$$

$$\partial_z \uparrow - \partial_y \uparrow = 0 \checkmark$$

$$\partial_z \uparrow - \partial_x \uparrow = 0 \checkmark$$

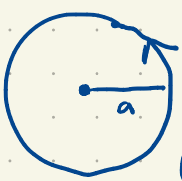
$$f = e^{xy} + h(y,z)$$
$$f_y = xe^{xy} + \partial_y h(y,z)$$

$$\partial_y h(y,z) = -\sin(z)$$
$$h(y,z) = -y\sin(z) + g(z)$$

$$f = e^{xy} - y\sin(z) + g(z)$$

$$\partial_z f = y\cos z + g'(z) \Rightarrow g(z) = \text{const.}$$

But that's not what curl is for



C

$$\underbrace{\frac{1}{2\pi a}\int_C \vec{V}\cdot d\vec{r}}_{}$$

→ avg speed in the ccw direction.

Distance around? $2\pi a$.

So fluid, on average traverses the circle in time

$$\left(\frac{1}{2\pi a}\right)^2 \int_C \vec{V}\cdot d\vec{r} = \left(\frac{1}{2\pi a}\right)^2 \iint_{\mathcal{E}} -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\, dA$$

$$= \frac{1}{4\pi}\,\frac{1}{\pi a^2}\iint_{\mathcal{E}} \uparrow dA$$

Let $a \to 0$ and circulation time is

$$\frac{1}{4\pi}\left[-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right] \quad \text{rotations / time}$$

$$\frac{1}{2}\left[-\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right] \quad \text{radians / time}$$

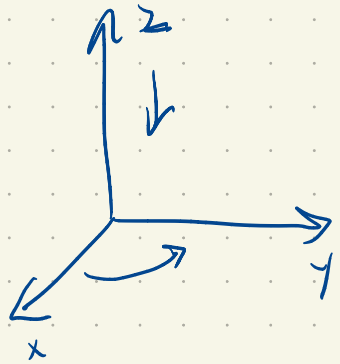
angular velocity in the x-y plane of the fluid.

Job of curl $\vec{X}$:

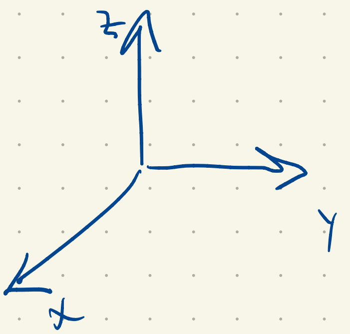
Pick a spot P and an normal vector $\vec{n}$ at P.

$\frac{1}{2}$ curl $\vec{X} \cdot \vec{n}$ tells you the circulation of the fluid in the plane perp to $\vec{n}$ in the ccw direction when looking from $\vec{n}$.

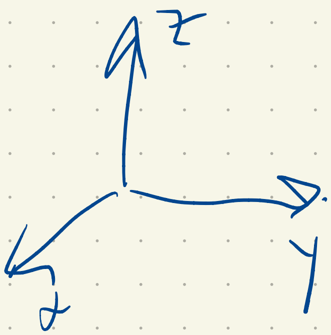
$\text{curl}\, X \cdot \hat{\imath} =$



$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}$$



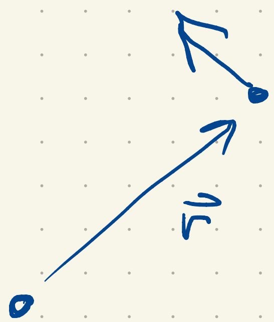
$$-\frac{\partial Q}{\partial z} + \frac{\partial R}{\partial y} = \text{curl}\, \vec{X} \cdot \hat{\jmath}$$



$$-\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z} = \text{curl}\, \vec{X} \cdot \hat{k}$$

The miracle: this works for any $\vec{n}$, not just $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

e.g. rotating fluid



$x = r\cos(\omega t)$

$y = r\sin(\omega t$

$x' = -r\sin(\omega t)\,\omega = -y\omega$

$y' = r\cos(\omega t)\,\omega = x\omega$

$\vec{V} = -y\omega\,\hat{\imath} + x\omega\,\hat{\jmath}$

$\vec{\nabla}\times\vec{V} = 2\omega\,\hat{k}$ everywhere!

$\vec{v} = \hat{\jmath}\, v_0\, e^{-x^2/\lambda^2}$

$-\frac{2 v_0 x}{\lambda} e^{-x^2/\lambda^2}\,\hat{k}$