Last class:

If $\vec{F} = P\hat{i} + Q\hat{j}$ is conservative

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$$

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But we saw an example

where $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0$ on its domain

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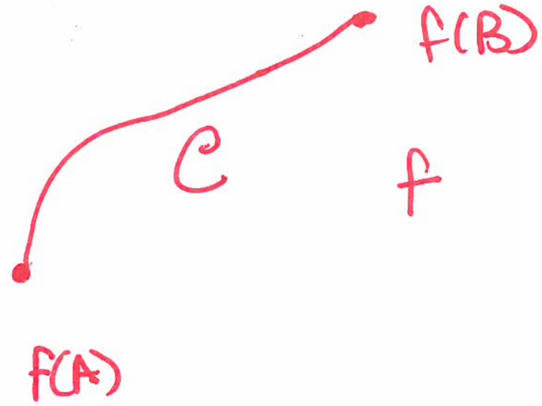
but $P\hat{i} + Q\hat{j}$ was not conservative,

Today we'll recover the situation

Green's Thm

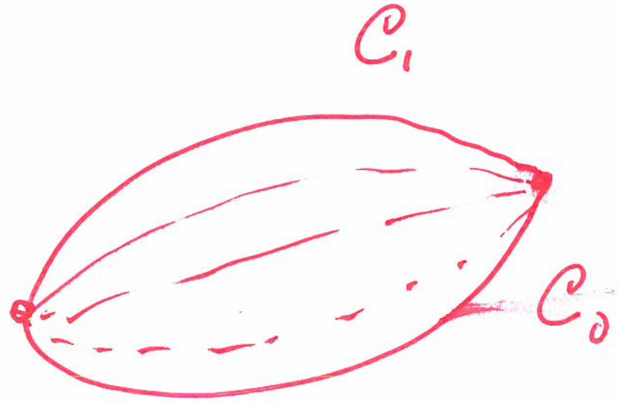



$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\int_C df = f(B) - f(A)$$

If you integrate up df in between, you get a net change in values.

$$\omega = P\,dx + Q\,dy$$




$C_s \rightarrow$ deform $C_0$ into $C_1$ while keeping endpoints fixed.

can we compute $\int_{C_1} \omega$ vs $\int_{C_2} \omega$

Shocking answer is that we can.

$$\omega = P\,dx + Q\,dy$$

$$d\omega = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dA$$

If $C_0$ and $C_1$ are oriented curves in the plane that start and end at the same points,

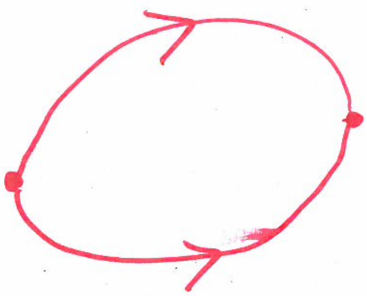
and if they have no point of intersection in between,

and if we can deform one into the other, then

$$\int_{C_1} \omega - \int_{C_0} \omega = \iint_E \left(-\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

(*) (orientation of interior

Alt: (Green)



$C$: do $C_0$, then $-C_1$

$C$ traversed counter-clockwise

$E$ has no holes

(simply connected)

Boundary has no self intersection.

$$\int_C P\,dx + Q\,dy = \iint_E \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dA$$

If you go other way round, integrand is

$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ instead. Sign change.

Consequences:

If $E$ is a simply connected domain

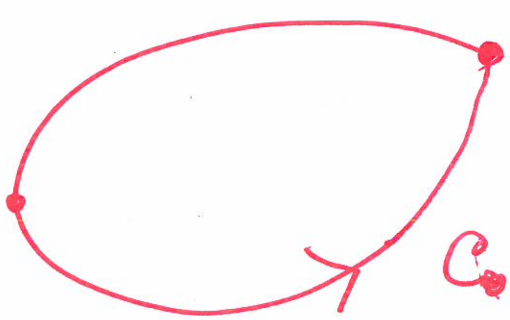
with a piecewise smooth boundary and

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = 0 \quad \text{on } E$$

then there exists $f$

$$P = \frac{\partial f}{\partial x} \qquad Q = \frac{\partial f}{\partial y}$$

So $P\hat{\imath} + Q\hat{\jmath}$ is conservative.

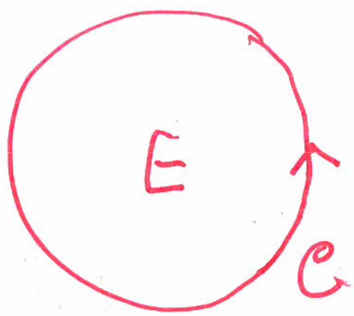


$C$

$$\int_C P\,dx + Q\,dy = \iint_E \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) =$$

e.g. $P = -x^2 y \qquad Q = x^3$

$E = \{ x^2 + y^2 < 4 \}$




$$\int_C P\,dx + Q\,dy = \iint_E -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\, dA$$

$\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$

$$\int_0^{2\pi} -\cos^2(t)\sin(t) + \cos^3(t)\, dt$$

$$\cancel{= \int_0^{2\pi} -\cos^3(t)\sin t + (1-\sin^2(t))\cos(t)\, dt}$$

$$\int_0^{2\pi} \cancel{4\cos^2 t(\sin t)\cdot(-2\sin(t)) + \cos}$$

$$2^4(\cos^2 t\sin^2 t) + 2^4\cos^3 t\sin t\, dt$$

$$\int_0^{2\pi} 2^4\cos^2 t - 2^4\sin^4(t))$$

$$\cancel{2^4((\cos^2 t}$$

$$2^2 \sin^2(2t)$$

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$$\frac{\partial P}{\partial y} = -x^2 \qquad \frac{\partial Q}{\partial x} = 3x^2$$

$$4x^2$$

$$\int_0^{2\pi}\int_0^2 4(r\cos\theta)^2 r\, dr\, d\theta$$

$$\int_0^{2\pi} \cos^2\theta\, r^4\Big|_0^2 d\theta = 2^4\int_0^{2\pi}\cos^2\theta$$

$$= 2^4\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\, d\theta$$

$$= 2^4\pi = 16\pi$$

$$P = -x^2 y \qquad Q = x^3$$

$$x = 2\cos t \qquad y = 2\sin(t)$$

$$x' = -2\sin t \qquad y' = 2\cos t$$

$$2^4 \int_0^{2\pi} +\cos^2 t \sin^2 t + \cos^3(t)\cos t \, dt$$

$$2^4 \int_0^{2\pi} \cos^2(t) \, dt = 2^4 \int_0^{2\pi} \frac{1+\cos(2t)}{2}$$

$$= 16\pi$$

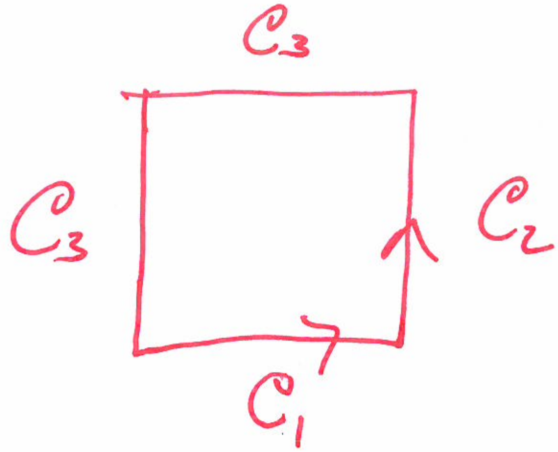
Shock: we can compute enclosed area from the boundary alone.

$$\int_{\mathcal{C}} -y\,dx = \iint_{\mathcal{E}} dA$$

$$\int_{\mathcal{C}} x\,dy = \iint_{\mathcal{E}} dA \qquad (!)$$

$$\int_{\mathcal{C}} \tfrac{1}{2}(-y\,dx + x\,dy) = \iint_{\mathcal{E}} dA.$$

$$\int_{C} P\,dx$$




$$\int_{C_1} P\,dx = \int_{x_0}^{x_1} P(x, y_0)\,dx$$

$$\int_{C_2} P\,dx = 0$$

$$\int_{C_3} P\,dx = \int_{x_1}^{x_0} P(x, y_1)\,dx = -\int_{x_0}^{x_1} P(x, y_1)\,dx$$

$$\int_{C} P\,dx = \int_{x_0}^{x_1} P(x, y_0) - P(x, y_1)\,dx$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\partial P}{\partial y}\,dy\,dx$$

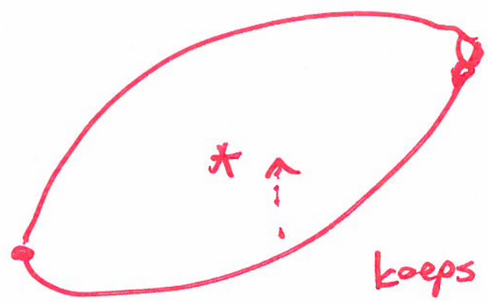
$$= \iint_{R} -\frac{\partial P}{\partial y}\,dA(x,y)$$

Note: If $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \equiv 0 \Rightarrow \int_{C_1} - \int_{C_2} P\,dx + Q\,dy = 0$

$\Rightarrow$ path ind

$\Rightarrow$ is conservative.

What's wrong with the holes?



keeps you from deforming one curve into another.