Last class:

$$\vec{F} = \vec{\nabla} f \qquad \int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$$

$$F = \langle M, N, P \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\int_C \underbrace{M\,dx + N\,dy + P\,dz} = f(B) - f(A)$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

Path independence for fixed endpoints

$$\Leftrightarrow \qquad \oint_C M\,dx + N\,dy + P\,dz = 0$$

for loops.

~~How can you tell if~~ $\cancel{M\,dx + N\,dy + P\,dz = df}$

$\cancel{M\hat{\imath} + N\hat{\jmath} + P\hat{k} = \vec{\nabla} f}$

$$\underbrace{M\,dx + N\,dy + P\,dz}_{\text{exact differential}} = df$$

$$\underbrace{M\hat{\imath} + N\hat{\jmath} + P\hat{k}}_{\text{conservative vector field}} = \vec{\nabla} f$$

See text: If $\displaystyle\int_C (M\,dx + N\,dy + P\,dz)$ $\left(\displaystyle\int_C \vec{F}\cdot d\vec{r}\right)$

depends only on endpoints,

then integrand is exact ($\vec{F}$ is cons.)

$$f(x,y,z) = \int_{C(x,y,z)} M\,dx \cdots + P\,dz = \int_C \vec{F}\cdot d\vec{r}$$

it is the work done to get from a ref point to (x,y,z), How can you tell if exact?

2-d: $\overset{P}{}dx + \overset{Q}{}dy = \underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}_{df}$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

If P, Q have cts 1st derivs

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

If exact $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

This can be used to rule out exactness

$$\vec{F} = \underbrace{(x^2+y)}_{P}\hat{i} + \underbrace{(2-xy)}_{Q}\hat{j}$$

$$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = -y \qquad 1 \neq -y \quad \text{done}$$

$$\vec{F} = \underbrace{e^x \cos y}_{P}\,\hat{i} \underbrace{- e^x \sin(y)}_{Q}\,\hat{j}$$

$$\frac{\partial P}{\partial y} = -e^x \sin y \qquad \frac{\partial Q}{\partial x} = -e^x \sin y \quad \checkmark$$

So who is the potential?

$$P = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = e^x \cos(y) + g(y)$$

$$\frac{\partial f}{\partial y} = -e^x \cos(y) + g'(y)$$

$$\Rightarrow g'(y) = 0 \Rightarrow g \text{ is const.}$$

$f(x,y) = -e^x \sin(y) + C$

$F = \underbrace{(3+2xy)}_{P}\hat{i} + \underbrace{(x^2-3y^2)}_{Q}\hat{j}$

$\frac{\partial P}{\partial y} = 2x \qquad \frac{\partial Q}{\partial x} = 2x \quad \checkmark$

$P = \frac{\partial f}{\partial x} \Rightarrow f = 3x + x^2y + g(y)$

$\frac{\partial f}{\partial y} = x^2 + g'(y)$

$g'(y) = -3y^2$

$g(y) = -y^3 + C$

$f(x,y) = 3x + x^2y - y^3 + C$

If $\vec{F}$ is conservative $\Rightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv 0.$$

What if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv 0$?

$$F = \frac{y}{\sqrt{x^2+y^2}}\,\hat{i} - \frac{x}{\sqrt{x^2+y^2}}\,\hat{j}$$

$$\cancel{\frac{\partial P}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} - \frac{2y}{(x^2+y^2)^2}}$$

$$F = \sqrt{x^2+y^2}\, y\, \hat{i} \ldots$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right)$$

$$\frac{-y}{x^2+y^2}\,\hat{i} + \frac{x}{x^2+y^2}\,\hat{j}$$

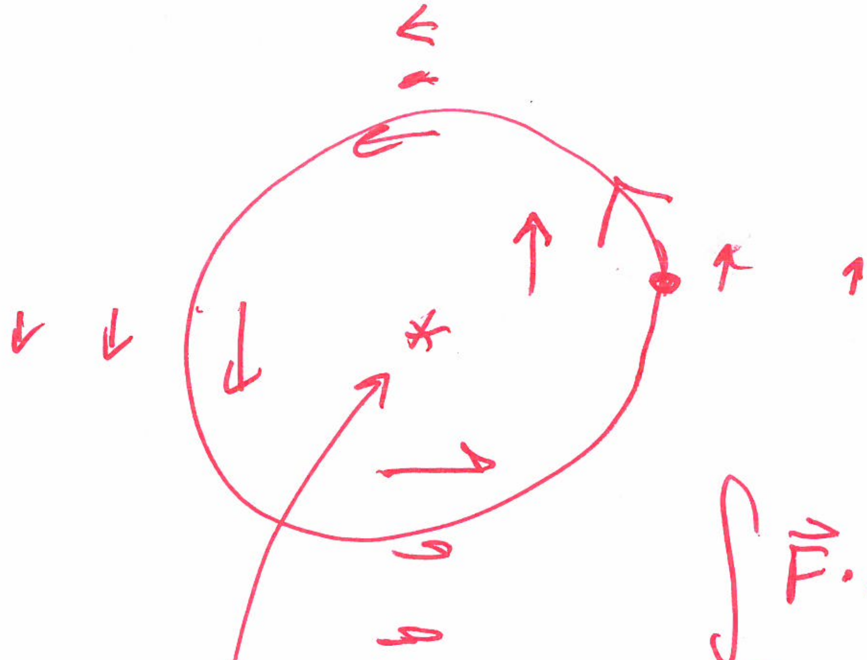
$$\frac{\partial P}{\partial y} = -\frac{1}{x^2+y^2} + \frac{y}{(x^2+y^2)^2}\cdot(2y)$$

$$= \frac{-x^2-y^2}{(x^2+y^2)^2} + \frac{2y^2}{(\text{—})^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial x} = \frac{1}{x^2+y^2} - \frac{x}{(x^2+y^2)^2}(2x)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$



$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} \neq 0 \quad (!)$$

the problem is here!

$$\Theta = \arctan\left(\frac{y}{x}\right) \quad (x>0)$$

$$d\theta = \frac{-x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy$$

$$\vec{\nabla}\theta = \uparrow \hat{\imath} + \uparrow \hat{\jmath}$$