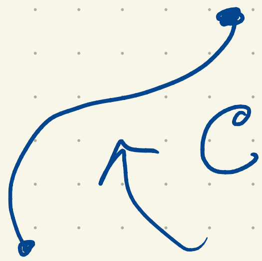
Line Integrals

a) Integrals with respect to arclength.



$\mathcal{C} \longrightarrow$ a curve

(not a parameterized curve

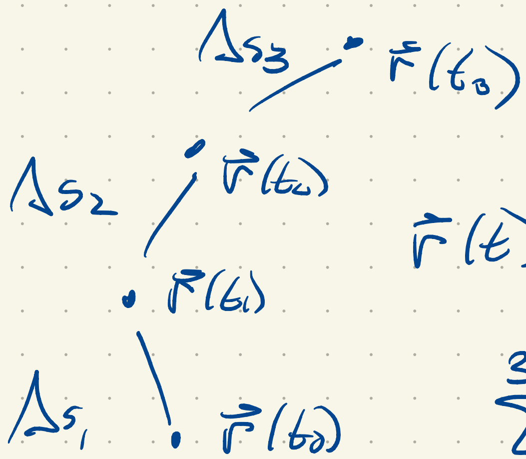
q: Given a function f defined along $\mathcal{C}$,

can we define the integral of f along $\mathcal{C}$?

Notation: $\displaystyle\int_{\mathcal{C}} f(\,?\,)\, d?$

What if $f \equiv 1$? $\displaystyle\int_{\mathcal{C}} 1 \; d?$

A reasonable choice: this is the length of $\mathcal{C}$.



$\vec{r}(t)$ a parameterization of $\mathcal{C}$

$$\underbrace{\sum_{k=1}^{3} \left|\vec{r}(t_k) - \vec{r}(t_{k-1})\right|}$$

$$\vec{r}(t_k) \approx \vec{r}(t_{k-1}) + \vec{r}\,'(t_{k-1})\Delta t_k$$

$$\Delta s_k \approx |\vec{r}\,'(t_{k-1})|\Delta t_k$$

To compute $\int_{\mathcal{C}} 1\, ds$, pick a parameterization, $\vec{r}(t)$

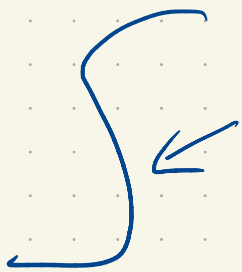
Then $$\int_{\mathcal{C}} ds = \int_{t_0}^{t_1} |\vec{r}\,'(t)|\, dt$$

ds is a thing you can integrate

and the above is a rule for how

to integrate it

Now suppose



$\rho(x,y,z)$ is density per arclength.

mass: $\approx \rho(x,y)\,\Delta s$

Total mass $\approx \displaystyle\sum_{k=1}^{n} \rho(x_k, y_k)\,\Delta s_k$

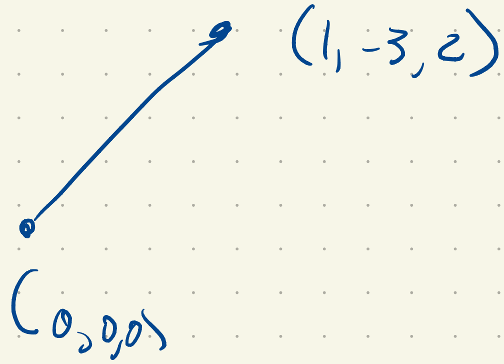
$$\int_C \rho(x,y)\,ds = \int_{t_0}^{t_1} \rho(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

for any parameterization of C,

eg:

$$\int_C (x+y^2-2z)\,ds$$

C:



$$\vec{r}(t) = \langle t, -3t, 2t \rangle \qquad 0 \le t \le 1$$

$$|\vec{r}\,'(t)| = \sqrt{1+9+4} = \sqrt{14}$$

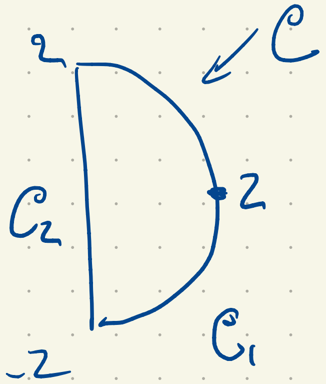
$$\int_0^1 \left(t + (-3t)^2 - 2(2t)\right)\sqrt{14}\,dt$$

$$\sqrt{14}\int_0^1 -3t + 9t^2\,dt = \sqrt{14}\left[-\frac{3t^2}{2} + \frac{9t^3}{3}\right]_0^1$$

$$= \sqrt{14}\left[-\frac{3}{2} + \frac{9}{3}\right]$$

$$= \frac{3\sqrt{14}}{2}$$

e.g.



$$\int_{\mathcal{C}} (1+xy)\,ds$$

$$\int_{\mathcal{C}_1} (1+xy)\,ds$$

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle \qquad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$|\vec{r}\,'(t)| = 2$$

$$\int_{-\frac{\pi}{2}}^{\pi/2} (1+4\cos(t)\sin(t))\,2\,dt$$

$$= 2\pi + 8\int_{-\pi/2}^{\pi/2} \cos(t)\sin(t)\,dt$$

$$= 2\pi + 8\int_{-1}^{1} u\,du \qquad \begin{array}{l} u = \sin(t) \\ du = \cos(t)\,dt \end{array}$$

$$= 2\pi + 8\frac{u^2}{2}\Big|_{-1}^{1} = 2\pi + 0 = 2\pi$$

$$\int_{C_2} (1+xy)\,ds = \int_0^4 (1+0)\,1 \cdot dt = 4$$

$\vec{r}(t) = \langle 0, 2-t \rangle \qquad 0 \le t \le 4$

$|\vec{r}\,'(t)| = 1$

$$\int_{C_2} (1+xy)\,ds = 4$$

$$\int_C (1+xy)\,ds = 2\pi + 4$$