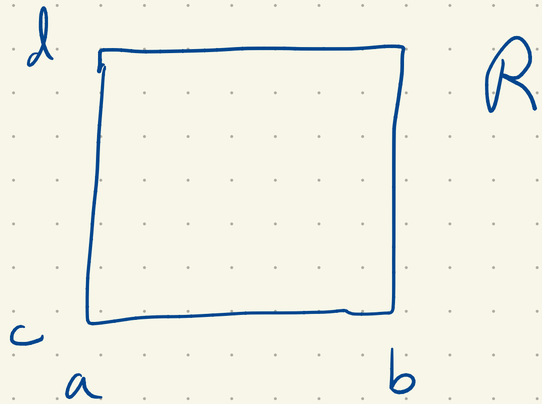
Section 15.2



$$\iint_R f(x,y)\,dA$$

$$=$$

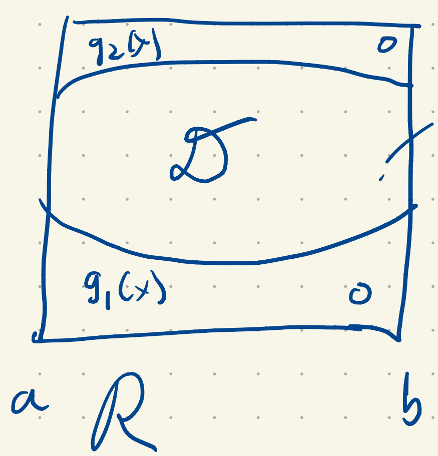
$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$=$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

If f is cts.

In fact it's true for a broader class of nice functions. Here are some

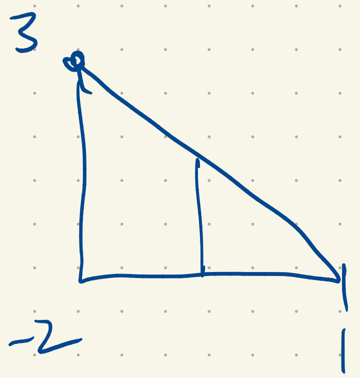


$$\iint_D f(x,y)\,dA = \iint_R f(x,y)\,dA$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

e.g. integrate $f(x,y) = 4-y$ in region bounded by

$x = -2$, $y = 0$, $y = 1-x$



$$\int_{-2}^{1} \int_{0}^{1-x} 4-y \, dy\, dx$$

$$= \int_{-2}^{1} \left(4y - \frac{y^2}{2}\right)\Big|_0^{(1-x)} dx$$

$$= \int_{-2}^{1} 4(1-x) - \frac{(1-x)^2}{2}\, dx$$

$$= 4x - \frac{4x^2}{2} + \frac{(1-x)^3}{6}\Big|_{-2}^{1}$$

$$= 4 - 2 - \left[-8 - 8 + \frac{27}{6}\right]$$

$$= 2 + 8 + 8 - \frac{9}{2}$$

$$= 18 - 4 - \frac{1}{2} = 14 - \frac{1}{2} = \frac{27}{2}$$

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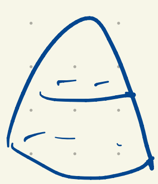
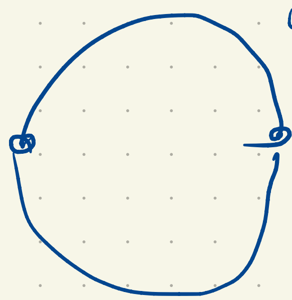
e.g. $z = 4 - x^2 - y^2$

Find region bounded by above and $z = 0$





$y = \sqrt{4-x^2}$

$x^2 + y^2 = 4$

$y = \pm\sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4 - x^2 - y^2 \, dy \, dx$$

$$\int_{-2}^{2}\left[\left(4y - x^2 y - \frac{y^3}{3}\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\right] dx$$

$$\int_{-2}^{2}\left[(4-x^2)\,2\sqrt{4-x^2} - \frac{2\left(\sqrt{4-x^2}\right)^3}{3}\right] dx$$

$$\int_{-2}^{2} \frac{4}{3}(4-x^2)\sqrt{4-x^2}\; dx$$

$$x = 2\sin\theta$$

$$dx = 2\cos\theta\, d\theta$$

$$\frac{4}{3}\int_{-2}^{2} (4-x^2)^{3/2}\, dx$$

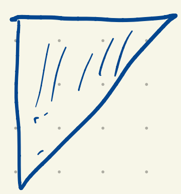
$$\frac{4}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2}\left(\cos^2\theta\right)^{3/2} 2\cos\theta\, d\theta$$

$$\frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{64}{3}\left[\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x\right]_{-\pi/2}^{\pi/2}$$

$$= 8\pi \quad (!)$$

$$\int_0^1 \int_x^1 \sin(y^2)\, dy \, dx$$



$$\int_0^1 \int_0^y \sin(y^2)\, dx \, dy = \int_0^1 y \sin(y^2)\, dy$$

$$u = y^2$$

$$du = 2y\, dy$$

$$= \frac{1}{2}\int_0^1 \sin(u)\, du$$

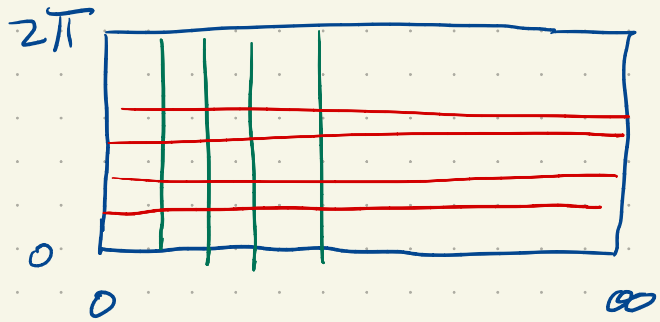
$$= \frac{1}{2}\left(-\cos(u)\right)\Big|_0^1$$

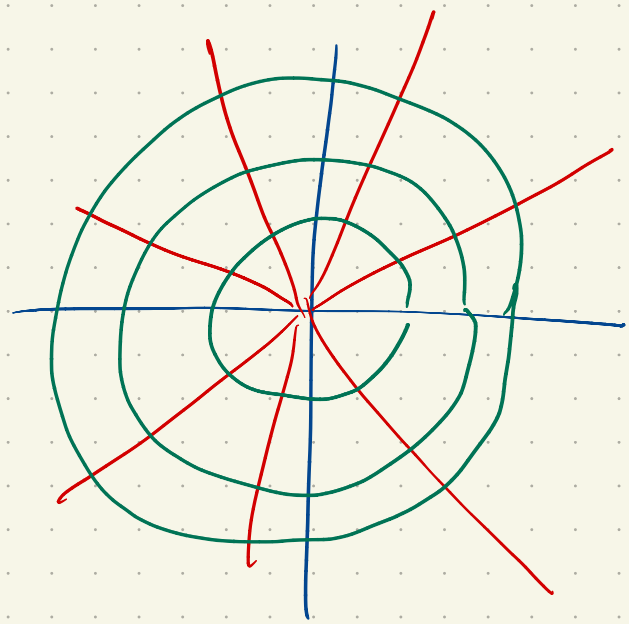
$$= \frac{1}{2}\left[-\cos(1) + \cos(0)\right]$$

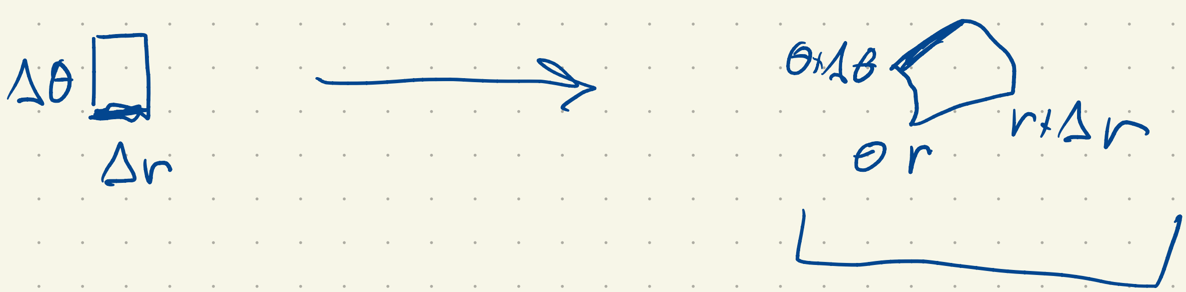
$$= \frac{1}{2}\left[1 - \cos(1)\right]$$

Section 15.3

Polar coordinates



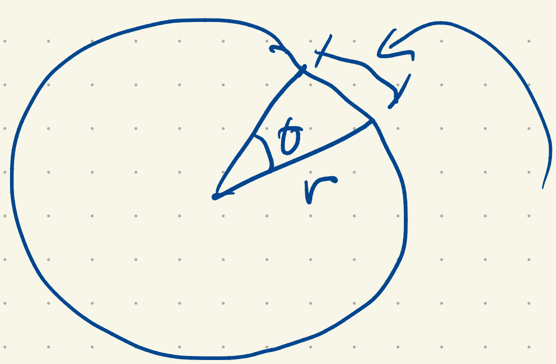




like a little rectangle

length Δr.

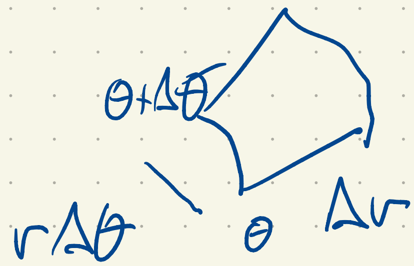
width? depends on r



total circumfrence $2\pi r$

part with angle θ

$$\frac{\theta}{2\pi} 2\pi r = \theta r$$

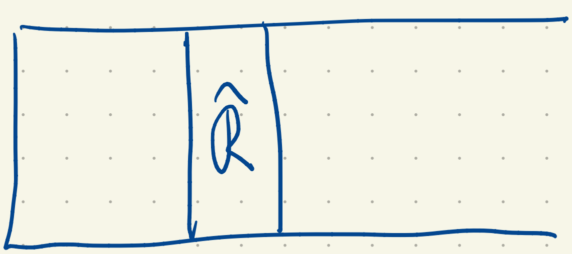


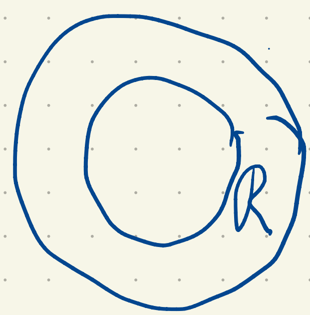
area is approximately

$$r\,\Delta r\,\Delta\theta$$

You want to integrate

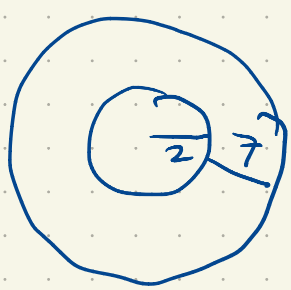
$$\iint_R f(x,y)\,dA$$





$$\iint_{\hat{R}} f(r\cos\theta, r\sin\theta)\,\underbrace{d\hat{A}}_{r\,dr\,d\theta} \qquad \iint_R f(x,y)\,\underbrace{dA}_{dx\,dy}$$

e.g.

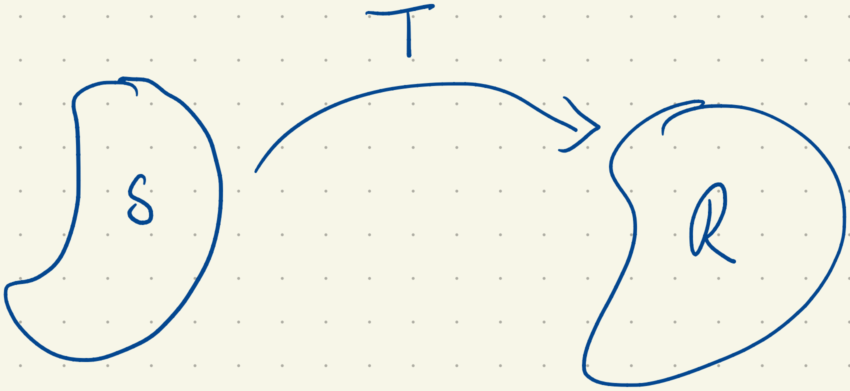


$$\int_0^{2\pi}\int_2^7 1\; r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_2^7$$

$$= \pi(7^2-2^2)$$

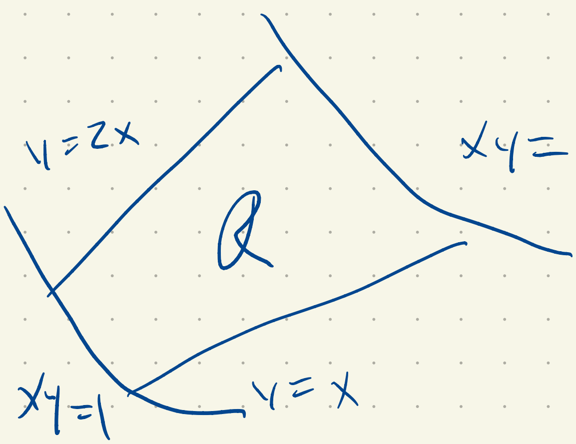
$$= \pi 45$$

$$dx\,dy = r\,dr\,d\theta$$



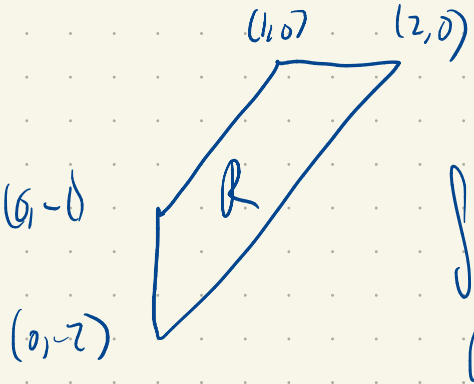
$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v)}$$

$$\int_{u(a)}^{u(b)} f(u)\,du = \int_a^b f(u(w))\underbrace{u'(w)}_{\frac{\partial u}{\partial w}}\,dw$$



$$\iint_R x^3 y\,dA$$

$$= \frac{13}{6}$$



$$\iint_R e^{x+y/(x-y)}\,dx\,dy$$

$$\begin{aligned} u &= x+y \\ v &= x-y \end{aligned} \longrightarrow \begin{aligned} \frac{u+v}{2} &= x \\ \frac{u-v}{2} &= y \end{aligned}$$

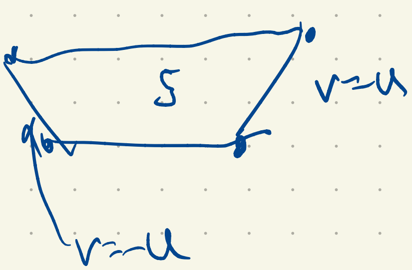
$$\frac{\partial x}{\partial u} = \frac{1}{2} \quad \frac{\partial x}{\partial v} = \frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} \quad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$J = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$(1,0) \to (1,1) \qquad (0,-1) \to (-1,1)$

$(2,0) \to (2,2)$



$$\iint_S e^{u/v}\,\frac{1}{2}\,du\,dv$$

$$\frac{1}{2}\int_1^2 \int_{-v}^{v} e^{u/v}\,du\,dv$$

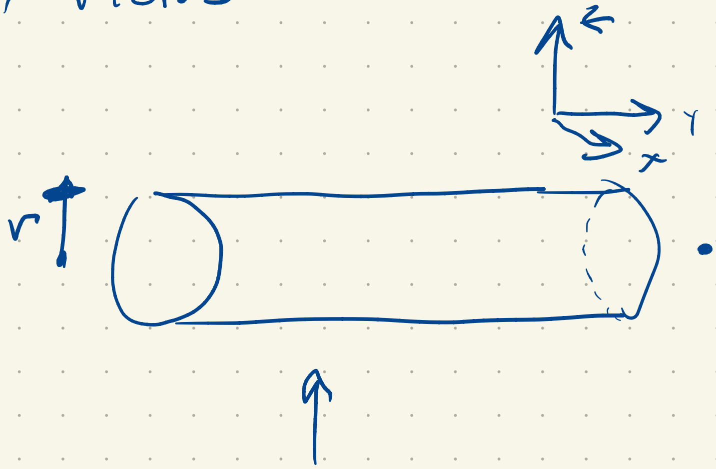
$$\frac{1}{2}\int_1^2 v e^{u/v}\Big|_{u=-v}^{v} dv$$

$$\frac{1}{2}\int_1^2 v\left[e-e^{-1}\right] dv$$

$$\frac{1}{2}\left(e^{-1}-e\right)\frac{v^2}{2}\Big|_1^2$$

$$\sinh(1)\left[2-\frac{1}{2}\right]=\frac{3}{4}\sinh(1)$$

Vector Fields



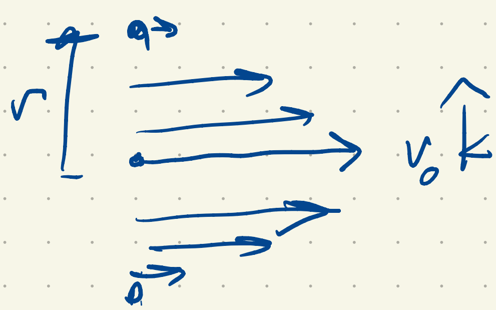
water in a pipe

Flow of water.

At each point, water has a velocity, a vector $\vec{V}$.

At the edge of the pipe, velocity is zero.

For "Poiseille" flow", velocity everywhere is in $\hat{k}$ direction, and has a parabolic profile



$$\vec{V}(x,y,z) = \frac{r^2 - x^2 - z^2}{r^2}\hat{k}$$

($\vec{v}$ is independent of y, variable down the pipe.

This is an example of a <u>vector</u> <u>field</u>.

In $\Omega \subseteq \mathbb{R}^2$, assigns each point a 2-d vector.

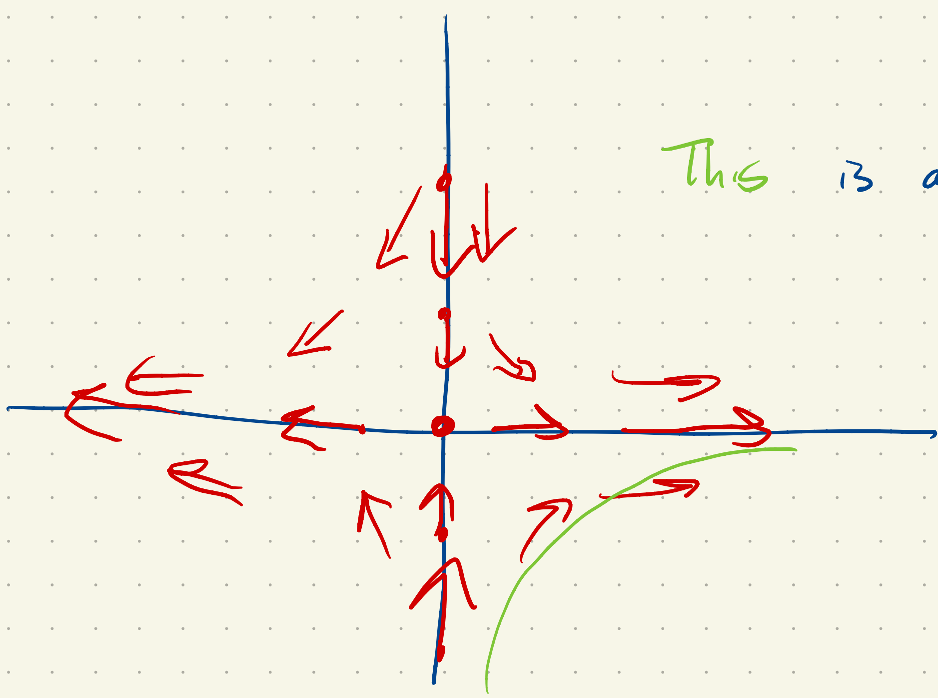
In $\Omega \subseteq \mathbb{R}^3$, assigns - - - 3-d vector

We've already seen these

$$F(x,y) = x^2 - y^2$$

$$\vec{\nabla} F = 2x\,\hat{\imath} - 2y\,\hat{\jmath}$$

$$(x,y) \longmapsto \vec{\nabla} F$$



This is again a vector field.

Here's another.

Electric charge: C

$e = -1.6 \times 10^{-19}$ C, e.g.

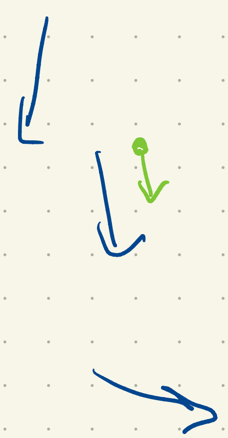
Electric field $\vec{E}$.

Job: A particle with charge q

experiences a force

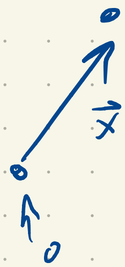
$$q\vec{E}$$

$$\frac{d}{dt}(mv) = q\vec{E}$$



You can't see it, but in principle $\vec{E}$ is everywhere.

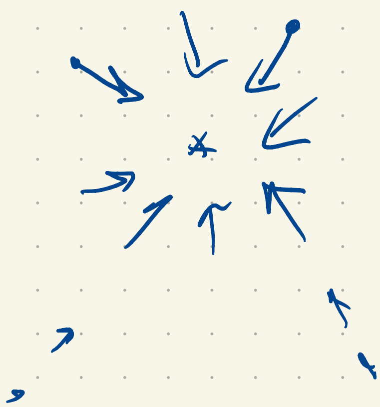
E.g: A stationary point particle with charge Q generates an electric field



$$\vec{E} = \frac{\vec{x}}{|\vec{x}|^3} Q$$

$$= \frac{Q}{|\vec{x}|^2} \underbrace{\frac{\vec{x}}{|\vec{x}|}}_{\text{unit vector}}$$

decays like $|\vec{x}|^2$



$Q < 0$

E.g. Gravity.

A point particle at $\vec{0}$ of mass M

Generates a grav field

$$\vec{g} = -\frac{GM\vec{x}}{|\vec{x}|^3} = -\frac{GM}{|\vec{x}|^2}\frac{\vec{x}}{|\vec{x}|}$$

$$[G] = \frac{[F][L^2]}{[m]^2}$$

$$\frac{[m][L]}{[T]^2}$$

$$\frac{[L^3]}{[T^2][m]}$$

and another particle with mass m

at position $\vec{x}$ experiences a force

due to gravity

$$6.67 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

$$\vec{F} = m\vec{g} = -$$

$$\frac{dy}{d\cancel{x}} = y$$

$y(t) = e^t$

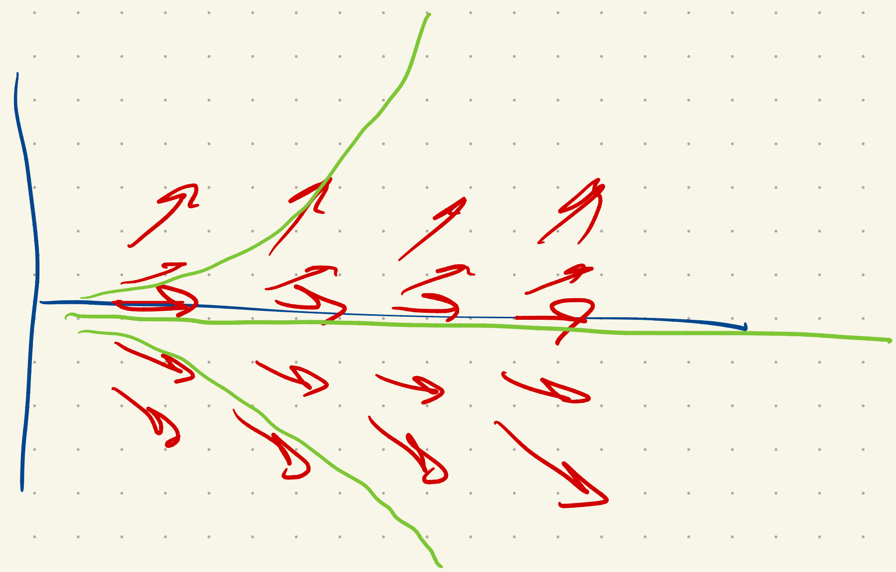
$y(t) = -2e^t$ are solutions

Associated vector field, called the direction field

$$\vec{F} = \underset{\substack{\uparrow \\ 1}}{} \hat{\imath} + y\hat{\jmath}$$

$\vec{r}(t) = \langle t, y(t) \rangle$

$\vec{r}\,'(t) = \langle 1, y'(t) \rangle = \langle 1, y \rangle$



A curve $\vec{r}(t)$ is called an <u>integral curve</u> of a vector field $\vec{F}$ if

for all t, $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$.

$\vec{r}(t) = \langle t, Ce^t \rangle$ $\qquad x(t) = t$

$\qquad y(t) = Ce^t$

$\vec{r}\,'(t) = \langle 1, Ce^t \rangle = \hat{\imath} + Ce^t \hat{\jmath} = \hat{\imath} + y(t)\hat{\jmath}$ ✓

$x(t) = t \qquad y(t) = Ce^t$

We have seen that the gradient provides an example of vector fields.

Def: A vector field $\vec{F}$ is <u>conservative</u> if $\exists$ a function f s.t.

$$\vec{F} = \vec{\nabla} f$$

in which case f is called a <u>potential function</u> for $\vec{F}$.

e.g. $f(\vec{x}) = \dfrac{1}{|\vec{x}|} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ in $\mathbb{R}^3 \setminus \{0\}$

$$\frac{\partial f}{\partial x} = -x\,(x^2+y^2+z^2)^{-3/2}$$

$$\vec{\nabla} f = \frac{-1}{|\vec{x}|^3}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$$

$$= -\frac{\vec{x}}{|\vec{x}|^3}$$

e.g., $f = -\frac{GM}{|\vec{x}|}$ is called the grav. potential

$V = \frac{Q}{|\vec{x}|}$ is called the electric potential

$$-\vec{\nabla} f = \vec{g}$$

$$-\vec{\nabla} V = \vec{E}$$

Does $y\hat{\imath} + 3x\hat{\jmath}$ have a potential?

$\frac{\partial f}{\partial x} = y$ $\qquad$ $\frac{\partial f}{\partial y} = 3x$

$\frac{\partial^2 f}{\partial x \partial y} = 1$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = 3$ $\qquad$ $1 \neq 3$, no.