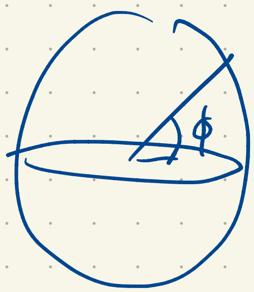
Alt:



$z = \rho \sin\phi$

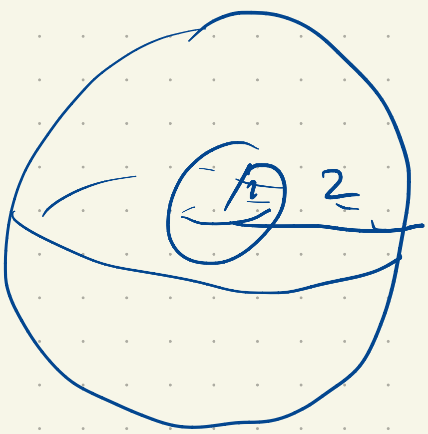
$x = \rho \cos\phi \cos\theta$

$y = \rho \cos\phi \sin\theta$

$$dV = \rho^2 \cos\phi \, d\rho \, d\theta \, d\phi$$

---

e.g.



$\mathcal{E}$

$$\iiint_{\mathcal{E}} z^2 \, dV$$

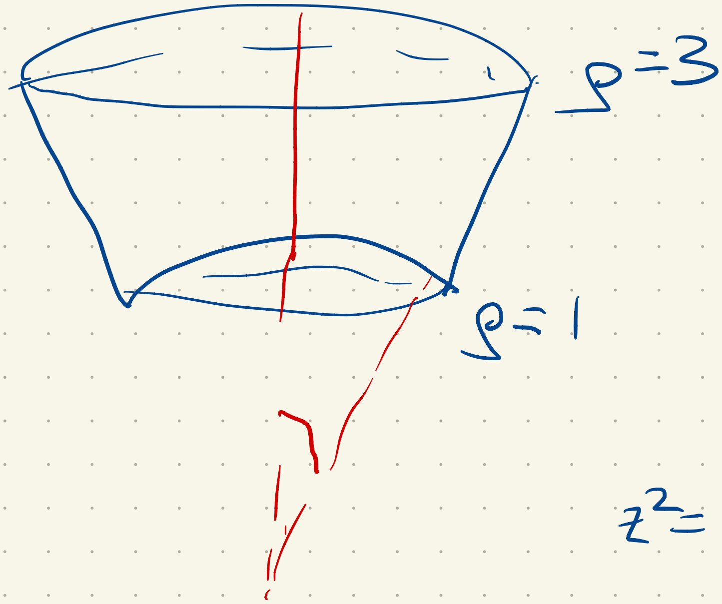
$$\int_0^{2\pi} \int_0^{\pi} \int_1^{2} \rho^2 \cos^2\phi \; \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi} \frac{\rho^5}{5}\Big|_1^2 \cos^2\phi \sin\phi \, d\phi \, d\theta$$

$$2\pi \left(\frac{2^5 - 1}{5} \right) \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \qquad u = \cos\phi$$

$$\boxed{\frac{124}{15}\pi}$$

e.g



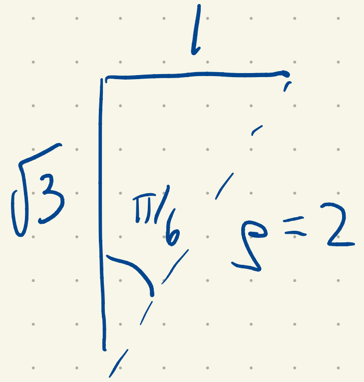
$z^2 = 3(x^2+y^2)$

$z = \sqrt{3}\, r$

Compute the volume.

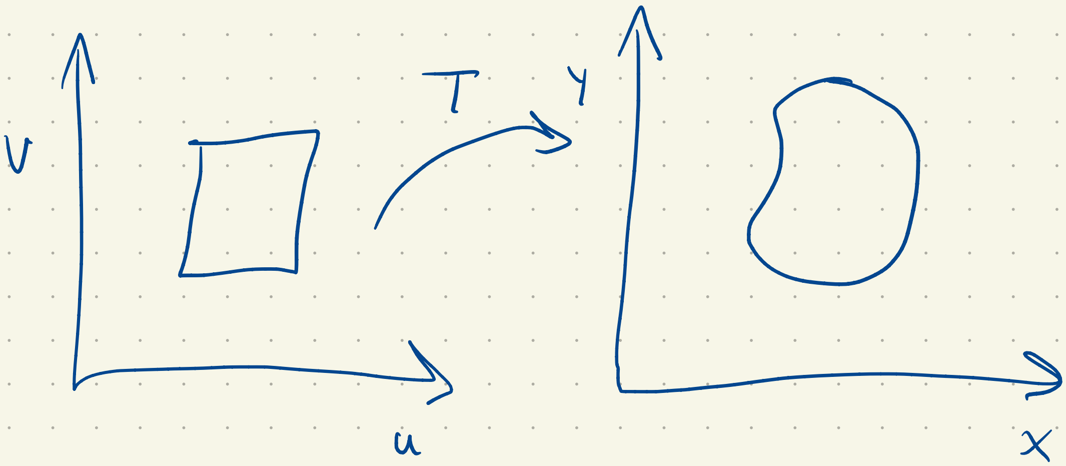
$\pi/6$

$$\iiint_E 1\, dV = \int_0^{2\pi} \int_1^3 \int_0^{?} 1 \sin\phi \, d\phi \, d\rho \, d\theta$$



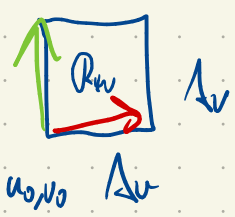
$$= \frac{26\pi}{3} \left(2 - \sqrt{3}\right)$$

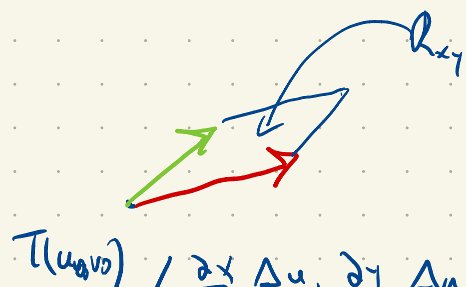
Change of variables



$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

We have an integral in x-y coords we want to express in u-v coords





$$\left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u \right\rangle$$

$$= \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u$$

green: $\left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v$

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \, \Delta v = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \, \Delta v$$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \longrightarrow \text{Jacobian matrix}$$

This is the real analog of the derivative.

Area of R_{xy} is related to the determinant of the matrix above, up to sign.

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \quad \text{Jacobian determinant.}$$

Moral: Area of $R_{uv} \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$

$$dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

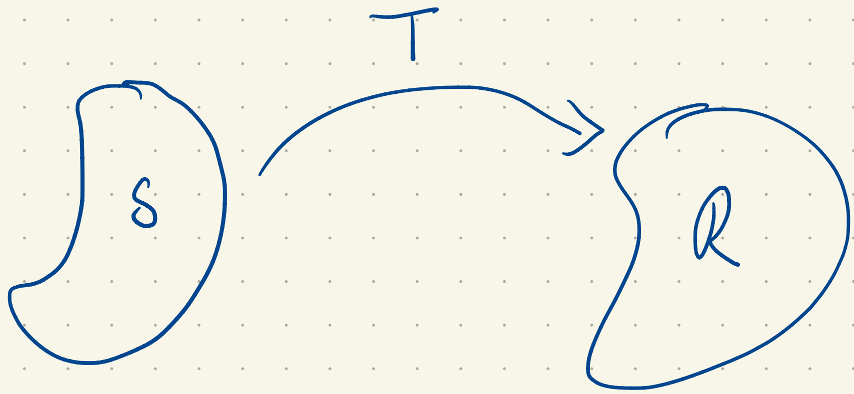
e.g. $x = r\cos\theta$

$y = r\sin\theta$

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

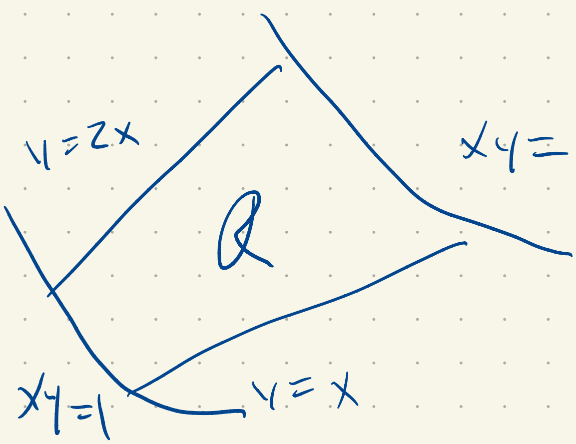
$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \left| r\cos^2\theta + r\sin^2\theta \right| = r \checkmark$$

$dx\,dy = r\,dr\,d\theta$

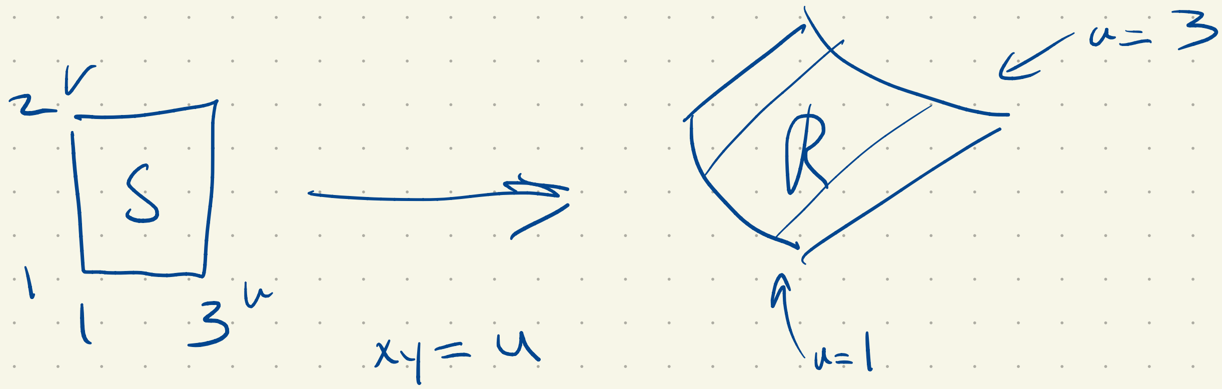


$$\boxed{\iint_R f(x,y)\,dA = \iint_S f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)}$$

$$\int_{u(a)}^{u(b)} f(u)\,du = \int_a^b f(u(w))\,\underbrace{u'(w)}_{\frac{\partial u}{\partial w}}\,dw$$



$$\iint_R x^3 y\,dA$$



$$xy = u$$

$$\frac{y}{x} = v$$

$$y = \sqrt{uv}$$

$$x = \sqrt{u/v}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{uv}} \qquad \frac{\partial x}{\partial v} = -\frac{1}{2}\sqrt{u}\, v^{-3/2}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}} \qquad \frac{\partial y}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}$$

$$J = \frac{1}{4}\left[\frac{1}{v} + \frac{1}{v}\right] = \frac{1}{2}\frac{1}{v}$$

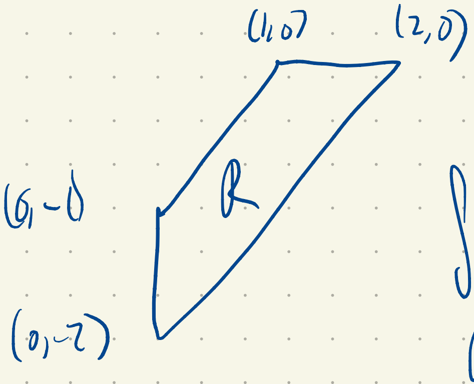
$$x^3 y = \frac{u^{3/2}}{v^{3/2}}\sqrt{u}\sqrt{v} = \frac{u^2}{v}$$

$$\int_1^2\int_1^3 \frac{u^2}{v}\,\frac{1}{2}\,\frac{1}{v}\,du\,dv = \int_1^2 \left.\frac{u^3}{6v^2}\right|_1^3 dv$$

$$= \frac{1}{6}[26]\int_1^2 \frac{1}{v^2}\,dv = \frac{26}{6}\left(-v^{-1}\right)\Big|_1^2$$

$$= \frac{26}{6}\left[-\frac{1}{2} + 1\right]$$

$$= \frac{13}{6}$$



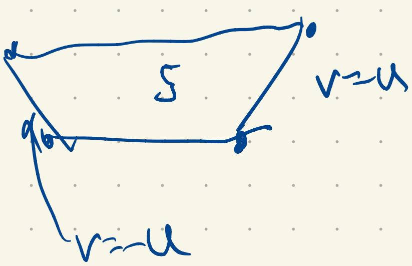
$$\iint_R e^{x+y/(x-y)}\, dx\,dy$$

$$u = x+y \longrightarrow \frac{u+v}{2} = x$$
$$v = x-y \qquad \frac{u-v}{2} = y$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \qquad \frac{\partial x}{\partial v} = \frac{1}{2}$$
$$\frac{\partial y}{\partial u} = \frac{1}{2} \qquad \frac{\partial y}{\partial v} = -\frac{1}{2}$$

$$J = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

$$(1,0) \to (1,1) \qquad (0,-1) \to (-1,1)$$
$$(2,0) \to (2,2)$$



$$\iint_S e^{u/v}\, \frac{1}{2}\, du\,dv$$

$$\frac{1}{2} \int_1^2 \int_{-v}^{v} e^{u/v}\, du\,dv$$

$$\frac{1}{2}\int_1^2 v e^{u/v} \Big|_{u=-v}^{u=v} dv$$

$$\frac{1}{2}\int_1^2 v \left[e - e^{-1} \right] dv$$

$$\frac{1}{2}\left(e^{-1} - e\right) \frac{v^2}{2}\Big|_1^2$$

$$\sinh(1)\left[2 - \frac{1}{2}\right] = \frac{3}{4}\sinh(1)$$