Section 15.6 Triple Integrals

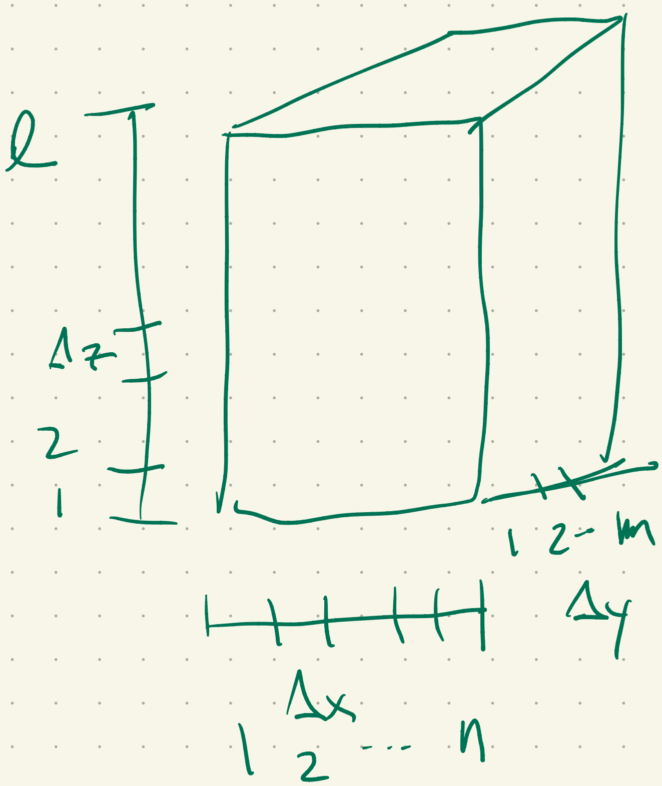
density of water: $1\, g/cm^3$

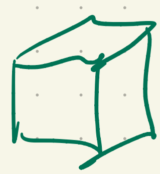
↑ volume

mass density at (x,y,z)

$$\approx \frac{\text{mass in sphere of radius } r}{\text{volume of region}}$$

improves as $V \to 0$.





← cube (i,j,k)

$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) = p^*_{ijk}$

in cube

mass of cube

$$\approx \rho(p^*_{ijk}) \cdot \Delta x \Delta y \Delta z$$

Approximate mass

$$\sum_{k=1}^{l} \sum_{j=1}^{m} \sum_{i=1}^{n} \rho(p^*_{ijk}) \Delta x \Delta y \Delta z$$

Riemann sum.

Now let $l, m, n \to \infty$. Results

$$\iiint_B \rho(x,y,z)\, dV$$

$B = [a,b] \times [c,d] \times [e,f]$

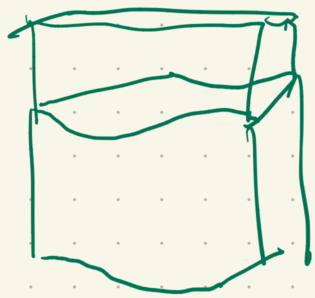
There is a Fubini's Theorem.

$$\int_a^b \int_c^d \int_e^f f(x,y,z)\, dz\, dy\, dx =$$

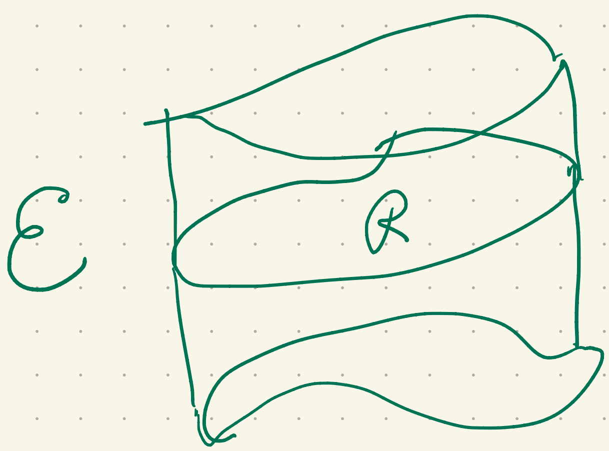
$$= \int_c^d \int_e^f \int_a^b f(x,y,z)\, dx\, dz\, dy = \ldots$$

This works if f is continuous

or if



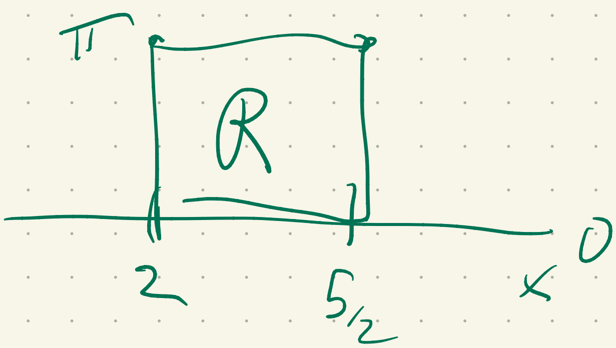
$$\int_a^b \int_c^d \int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

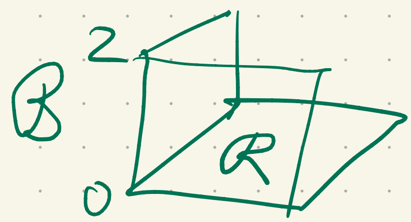


$$\iiint_{\mathcal{E}} f(x,y,z)\, dV = \iiint_{B} f(x,y,z)\, dV$$

$$= \iint_R \int_{F_1(x,y)}^{F_2(x,y)} f(x,y,z)\,dz\,dA$$

Now use double integral techniques.





Compute $\displaystyle\iiint_B zx\sin(xy)\,dV$

$$\int_0^2\int_2^{5/2}\int_0^{\pi} zx\sin(xy)\,dy\,dx\,dz$$

$u = xy$ (x const)

$du = x\,dy$

$$\int_0^2\int_2^{5/2}\int_0^{\pi x} z\sin(u)\,du\,dx\,dz$$

$$\int_0^2 \int_2^{5/2} z\,(-\cos u)\Big|_{u=0}^{\pi x} \, dx\, dz$$

$$\int_0^2 \int_2^{5/2} z\,\big(1-\cos(\pi x)\big)\, dx\, dz$$

$$\int_2^{5/2} \int_0^2 z\,\big(\underline{\hspace{3cm}}\big)\, dz\, dx$$

$$\int_2^{5/2} 2\,\big(1-\cos(\pi x)\big)\, dx$$

$$2\left(5/2-2\right) - \frac{2}{\pi}\sin(\pi x)\Big|_2^{5/2}$$

$$1-\frac{2}{\pi} \quad \checkmark$$

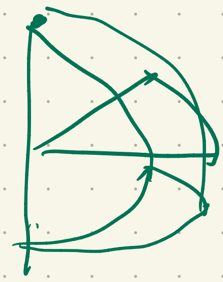
Integrate $\iiint\limits_E y\,dV$

e.g. $x \geq 0$

$y \geq 0$

$z = 3 - x^2 - y^2 \qquad z = -5 + x^2 + y^2$

$4 = x^2 + y^2$ is boundary



$$\iint\limits_R \int_{-5+x^2+y^2}^{3-x^2-y^2} y \, dz \, dA$$

$$\iint\limits_R y\left[3 - x^2 - y^2 + 5 - x^2 - y^2\right] dA$$

$$\iint\limits_R y\left[8 - 2(x^2+y^2)\right] dA$$

$$\int_0^{\pi/2}\int_0^2 r\sin\theta\,[8-2r^2]\,r\,dr\,d\theta$$

$$\int_0^2\int_0^{\pi/2} \sin\theta\,[8r^2-2r^4]\,d\theta\,dr$$

$$\int_0^2 -\cos\theta\Big|_0^{\pi/2}\,[\quad]\,dr$$

$$\int_0^2 (8r^2-2r^4)\,dr$$

$$\frac{8r^3}{3}-\frac{2r^5}{5}\Big|_0^2$$

$$20-6=14$$

$$\frac{64}{3}-\frac{64}{5}$$

$$\boxed{\frac{128}{15}}$$

To compute a volume integrate 1.

$$\iiint_{\mathcal{E}} 1 \, dV = \text{vol}(\mathcal{E})$$

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e.g. Chose density $\rho(x,y,z)$ is dist from xy plane

$|z|$

Region: sphere of radius 1

$$\iint_R \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z \, dz \, dA$$

$$\iint_R \int_0^{\sqrt{1-x^2-y^2}} z - \iint_R \int_{-\sqrt{1-x^2-y^2}}^{0} -z \, dz \, dA$$

$$2 \iint_R \frac{1}{2} z^2 \Big|_{z=0}^{\sqrt{1-x^2-y^2}} dA$$

$$\iint_R (1-r^2)\, dA$$

$$\int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]\Bigg|_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{4} \right)$$

$$= \frac{\pi}{2} \quad (!)$$