What's on the exam??

Chapter 14

15.1 – 15.4

What are the big topics:

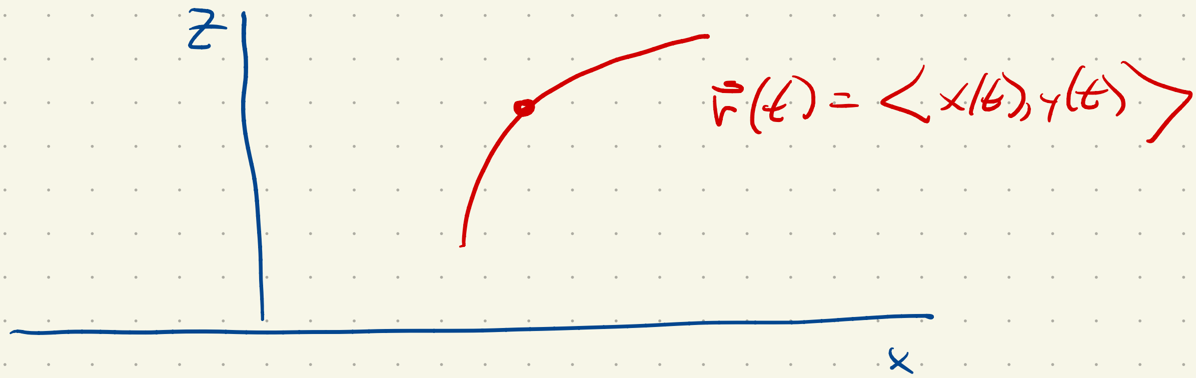
- limits are complicated
- partial derivatives
- gradient
- linear approx, differentials
- chain rule
- critical points, optimization
- use D to distinguish max/min/saddle

- set up a double integral vs an iterated integral
- computation of an iterated integral by reversing order
- set up an integral in polar coordinates
- computation of
  a) moments of mass
  b) centroids
  c) moments of inertia
  d) probabilities
  e) areas (2-d)
  f) volumes (3-d)
  g) mass from density
  h) foo from foo density

Common quiz errors

Suppose I tell you

$P(x,z)$ pressure



$$\frac{d}{dt} P(x(t), y(t)) = \underbrace{\frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt}}_{\text{chain rule}}$$

$$= \underbrace{\vec{\nabla} P \cdot \vec{r}\,'(t)}$$

↳ this is why we care about the gradient.

$\vec{\nabla} P$

What if I only let you travel at unit speed, $|\vec{r}\,'|=1$.

$$\vec{\nabla} P \cdot \vec{r}\,' = |\vec{\nabla} P|\,|\vec{r}\,'| \cos\theta$$

$$= |\vec{\nabla} P| \cos\theta.$$

So $|\vec{\nabla} P|$ is the biggest rate of change you see when travelling at unit speed and you see it when $\cos\theta = 0 \Rightarrow \theta = 0.$

$$\frac{d}{dt} T(\vec{r}(t))$$

$\vec{r}(0) = (2,3)$ $\quad \vec{r}\,'(0) = -$

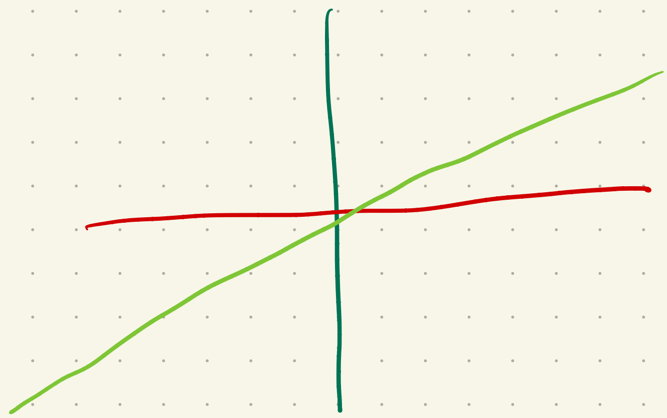
$\left.\frac{\partial T}{\partial x}\right|_{(2,3)} = 5 \qquad \left.\frac{\partial T}{\partial y}\right|_{(2,3)} = 7$

etc.

Limits:

$$\lim_{(x,y)\to(0,0)} f(x,y) = L \quad \text{means what?}$$

$$\frac{xy}{x^2+y^2}$$



$y = mx$

$$\frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2} \quad (x \neq 0)$$

$\uparrow$

$m = 0 \Rightarrow 0$

$m = 1 \Rightarrow 1/2$

$m = 3 \Rightarrow 3/10$

$m = \infty \Rightarrow 0$

$m = 1/3 \Rightarrow \frac{1}{3} \cdot \frac{1}{1+\left(\frac{1}{3}\right)^2}$

$\Rightarrow \frac{3}{10}$

Optimization Models

$f(x,y) = x^2 + y^2$

$\vec{\nabla} f = 0$ at $(0,0)$

in all cases

$f(x,y) = -x^2 - y^2$

$f(x,y) = x^2 - y^2$

$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $D = 4$

$H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $D = 4$

can't tell max from min

$H = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $D = -4$

$f(x,y) = x^4 + y^4$

$f(x,y) = -x^4 - y^4$

$f(x,y) = x^4 - y^4$

$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ at $\vec{0}$

in all cases

$D = 0$ is indeterminate

$$f(x, y) = 2x^2 + y^4 - 4xy$$

$\langle 4x - 4y,\ 4y^3 - 4x \rangle$

$4x - 4y = 0 \Rightarrow x = y$

$4y^3 - 4x = 0$ and $x = y \Rightarrow$

$4(y^3 - y) = 0$

$$4(y^2-1)y = 0$$

$$y = 0, \quad y = \pm 1.$$

Just 3 points ($y = x$ is a condition)

3. (7 pts.) Give an equation for the tangent plane to the surface $x^2 - 2y^2 + z^2 + yz = 2$ at the point $(2, 1, -1)$.

$F(x,y,z) =$

$F_x = 2x$

$F_y = -4y + z$

$F_z = y$

$\vec{\nabla} F = \langle 4, -5, 1 \rangle$

$4(x-2) - 5(y-1) + 1(z+1) = 0$

4. (15 pts.–5 pts. each) The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in $^\circ C$, and x, y, z are measured in meters.

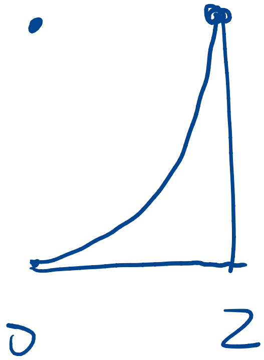
(a) Find the rate of change of the temperature at the point $(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$. GIVE UNITS.

(b) At $(2, -1, 2)$, in what direction does T increase most rapidly?

(c) What is the maximum rate of change of T at $(2, -1, 2)$, among all directions?

5. (12 pts.) Reverse the order of integration in the following integral, and then evaluate it.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$$



$x = \sqrt{y} \Rightarrow y = x^2$

$$\int_0^2 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$\int_0^2 \sqrt{x^3+1}\, y \Big|_{y=0}^{y=x^2} dx = \frac{52}{9}$$

6. (10 pts.) Ohm's law states that in an electrical circuit the current, I, depends on the voltage, V, and resistance, R, by

$$I = V/R.$$

Suppose at some moment $R = 100$ ohms, $V = 32$ volts, $dR/dt = 0.03$ ohms/s, and $dV/dt = -0.01$ volts/s. Determine dI/dt at that moment. GIVE UNITS. (Hint: Use the multivariable chain rule. The unit 'volt/ohm' is also called an 'ampere'.)

7. (12 pts.) Use the method of Lagrange multipliers to find the point on the sphere $x^2 + y^2 + z^2 = 70$ that minimizes $f(x, y, z) = 2x + 6y + 10z$.

8. (12 pts.–3 pts. each) Complete the following.

   (a) The average value of a function $f(x, y)$ over a 2-dimensional region R is given by the formula:

   $\frac{1}{\text{area}(R)} \iint_R f$

   (b) In spherical coordinates, dV is:

   $r^2 \, dr \, d\theta \, d\phi$

   (c) $\lim_{(x,y)\to(0,0)} \frac{2x^2 - xy^2}{x^2 + y^2}$ does not exist since:

   $y = mx$

   (d) The geometric relationship between the level curves of a function $z = f(x, y)$ and the gradient vectors $\nabla f(x, y)$ is:

   ortho