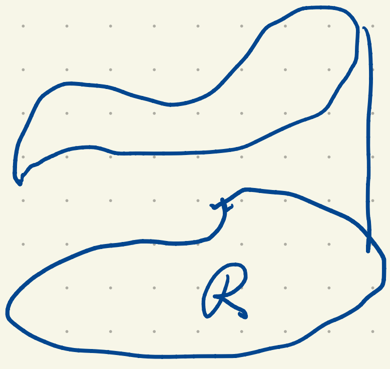
Section 15.4 Applications

1) 2-d areas

$$\iint_R 1 \, dA(x,y) = \text{area}$$

2) 3-d volumes



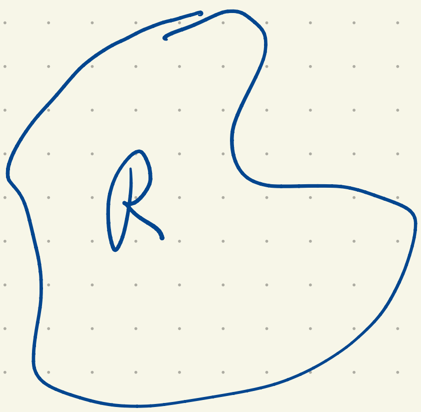
$$z = f(x,y) \geq 0$$

$$\iint_R f(x,y) \, dA(x,y) = \text{volume}$$

$\left(f(x,y) = 1, \text{ area} \cdot \text{height} = \text{area!} \right)$

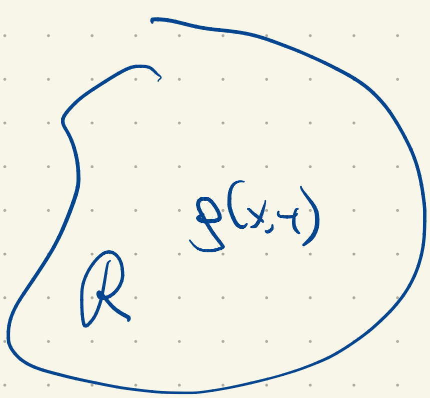
3) Averages

$T(x,y)$



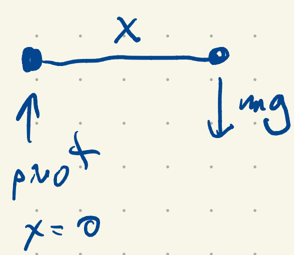
$$\frac{1}{\text{area}(R)} \iint_R T(x,y)\, dA(x,y)$$

4) density $\longrightarrow$ total mass

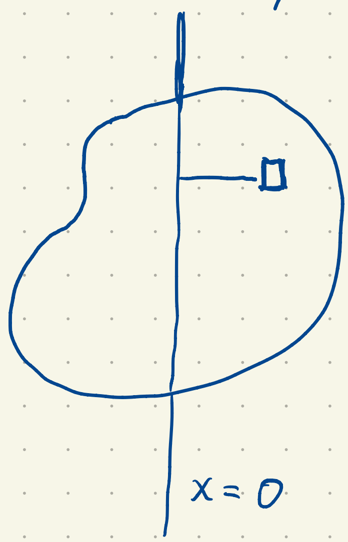


$$\iint_R \rho(x,y)\, dA(x,y) = \text{mass}$$

Moments



torque: $x\,mg$ down



contribution to torque

$$\hat{x}_i \; \rho(\hat{x}_i, \hat{y}_j)\, \Delta x\, \Delta y \cdot g$$

$$\sum_{i,j} g\, \hat{x}_i\, \rho(\hat{x}_i, \hat{y}_j)\, \Delta x\, \Delta y$$

$$\iint_R g\, x\, \rho(x, y)\, dA(x,y)$$

Important subcase: ρ constant.

$$M_y = \iint_R x \overset{\rho(x,y)}{\,} dA(x,y) \rightarrow$$

moment about y-axis
$x = 0$

$$M_x = \iint_R \underset{\uparrow \, \rho(x,y)}{y} \, dA(x,y) \rightarrow \text{moment about x-axis } (y=0)$$

Total mass: m

$$\left.\begin{aligned} m\bar{x} &= M_y \\ m\bar{y} &= M_x \end{aligned}\right] \text{ centroid } (\bar{x}, \bar{y})$$

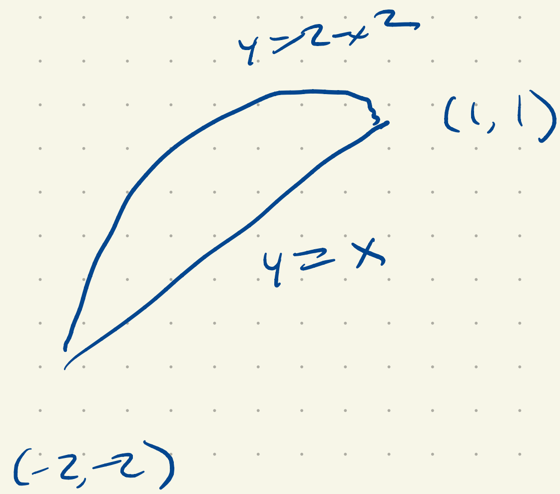
For constant density: $\bar{x} = \dfrac{1}{\text{area}(R)} \iint_R x \, dA$

its the average value of x.

$$\bar{y} = \frac{1}{\text{area}(R)} \iint_R y \, dA.$$

E.s. Assume $\rho = 1$.

Compute moments of



$$M_y = \int_{-2}^{1} \int_{x}^{2-x^2} x \, dy \, dx = \int_{-2}^{1} x\left(2 - x^2 - x\right) dx$$

$$= \int_{-2}^{1} 2x - x^3 - x^2 \, dx$$

$$= \frac{2x^2}{2} - \frac{x^4}{4} - \frac{x^3}{3} \Big|_{-2}^{1}$$

$$= -\frac{9}{4}$$

$$M_x = \int_{-2}^{1} \int_{x}^{2-x^2} y \, dy \, dx = \int_{-2}^{1} \frac{y^2}{2} \Big|_{x}^{2-x^2} dx$$

$$= \int_{-2}^{1} \frac{(2-x^2)^2}{2} - \frac{x^2}{2} \, dx$$

$$= \int_{-2}^{1} \frac{4-4x^2+x^4}{2} - \frac{x^2}{2} dx$$

$$= \int_{-2}^{1} 2 - \frac{5x^2}{2} + \frac{x^4}{2} \, dx$$

$$= 2x - \frac{5x^3}{6} + \frac{x^5}{10} \Big|_{-2}^{1}$$

$$= \frac{9}{5}$$

Finally: area:

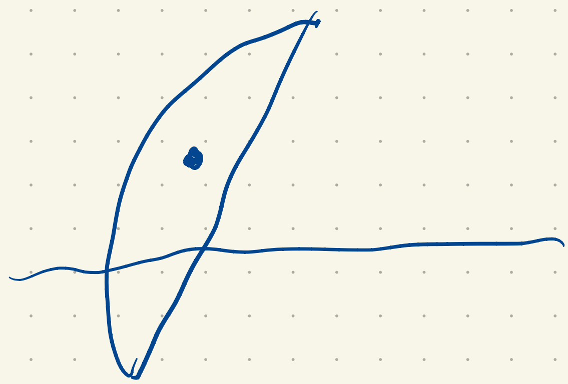
$$\int_{-2}^{1} \int_{x}^{2-x^2} 1 \, dy \, dx = \int_{-2}^{1} 2-x^2-x \, dx$$

$$= 2x - \frac{x^3}{3} - \frac{x^2}{2} \Big|_{-2}^{1}$$

$$= 9/2$$

$$\bar{x} = \frac{M_y}{m} = \frac{-9/4}{9/2} = -\frac{1}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{9/5}{9/2} = \frac{2}{5}$$



$$\iint (x-a)\, dA = 0$$

$$\iint x = a \iint dA$$

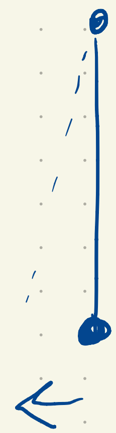
$$A \cdot \bar{x} = a\, A$$

$$\bar{x} = a$$

$\tau = I\,\dot{\omega}$

$\frac{1}{s^2}$

$\frac{LML}{s^2}$ $\quad [I] = ML^2$



$m\ddot{x} = F$

$ml\ddot{x} = \tau$

$x = r\cos\theta$

$\dot{x} = r\cos\theta\,\dot{\theta}$

$\ddot{x} = r\cos\theta\,\ddot{\theta} + O$

m

2nd moment $\qquad \tau = I\dot{\omega} \qquad F = m\dot{v}$

$$I_y = \iint_R \rho(x,y)\, x^2 \, dA$$ (moment about y-axis)

$$I_x = \iint_R \rho(x,y)\, y^2 \, dA$$

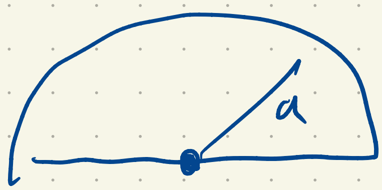
polar moment

$$I_0 \quad \iint_R \rho(x,y)\; x^2+y^2 \, dA = I_x + I_y \quad (!)$$

(resistance to spinning

about z-axis)

E.g. 2^{nd} moment of a $\frac{1}{2}$ disk $\rho(r) = \rho$ $r = a$



$$I_x = \iint_R \rho y^2 \, dA$$

$$= \rho \iint_R y^2 \, dA$$

$$= \rho \int_0^{\pi} \int_0^{a} r^3 \sin^2\theta \, dr \, d\theta$$

$$= \rho \int_0^{\pi} \frac{a^4}{4} \sin^2\theta \, d\theta$$

$$= \frac{\rho a^4}{4} \int_0^{\pi} \sin^2\theta \, d\theta$$

$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$

$$= \frac{\rho a^4}{3} \int_0^{\pi} \frac{1-\cos 2\theta}{2} \, d\theta$$

$$= \frac{\rho a^3}{4} \left[\frac{\theta - \sin 2\theta}{2} \right]_0^{\pi}$$

$$= \frac{\pi}{2} \frac{\rho a^{\cancel{3}4}}{\cancel{3}4}$$

$$\int_0^{\pi} \int_0^{a} \rho r^2 \cos^2\theta \, r \, dr \, d\theta = \frac{\rho a^{\cancel{4}}}{3} \int_0^{\pi} \cos^2\theta \, d\theta$$

$$= \frac{\rho a^{\cancel{4}}}{3} \left[\int_0^{\pi} \frac{1}{2} \left[1 + \cos 2\theta \right] \right]$$

$$= \frac{\rho a^{\cancel{4}}}{3} \frac{\pi}{\cancel{2}4} \quad (!)$$