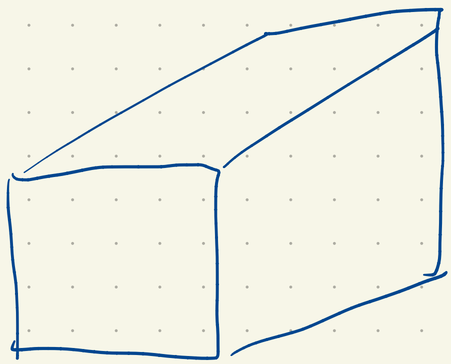
Section Lagrange Multipliers



$V = xyz$

girth + length ≤ 108

$2x+2y+z \leq 108$

clearly an increase, so

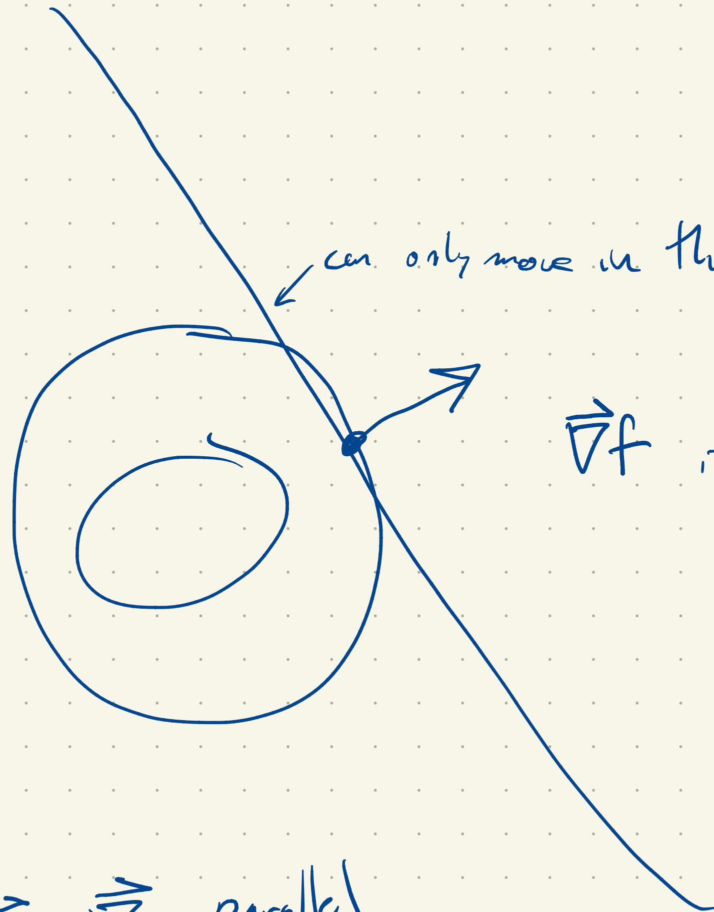
Maximize $V=xyz$ subject to

$$\underbrace{2x+2y+z=108}_{\text{constraint}}$$

We'll come back to this.

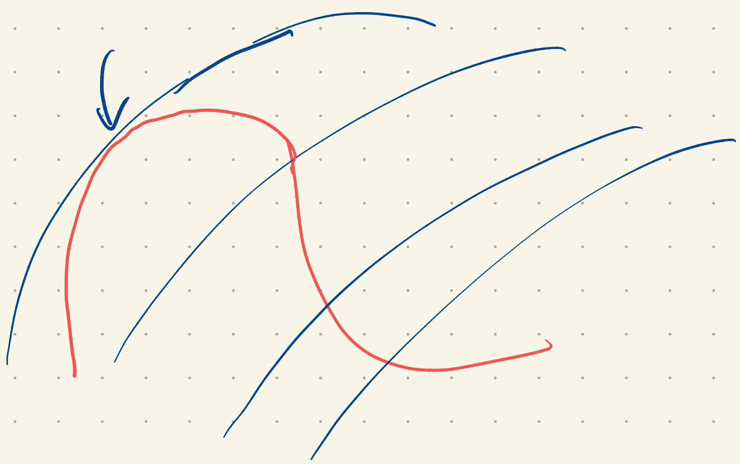
Let us instead minimize

$f(x,y) = x^2+y^2$ subject to $\underbrace{x+y}_{g(x,y)} = 9$



$\vec{\nabla} f$ is parallel to $\vec{\nabla} g$

$\vec{\nabla} f, \vec{\nabla} g$ parallel



At a maximizer

$$\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$$

3 eq's for 3 unknowns.

$$\begin{cases} g(x_0,y_0) = 9 & (c \text{ in general}) \\ f_x(x_0,y_0) = \lambda g_x(x_0,y_0) \\ f_y(x_0,y_0) = \lambda g_y(x_0,y_0) \end{cases}$$

$\vec{\nabla} f = \langle 2x, 2y \rangle$

$\vec{\nabla} g = \langle 1, 1 \rangle$

$x+y = 9$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \end{aligned}\right] \quad x = y = \frac{\lambda}{2}$$

$x + x = 9 \Rightarrow \boxed{x = 9/2 \ , \ y = 9/2}$

($\lambda = 9$ is not essential)

$f(9/2, 9/2) = \frac{81}{4} \cdot 2 = \frac{81}{2}$

---

e.g. Find extreme values of

$x^2 + 4y^3$ on the ellipse $x^2 + 2y^2 = 1$

$\nabla f = \langle 2x, 12y^2 \rangle$

$\nabla g = \langle 2x, 4y \rangle$

$2x = \lambda 2x$

$12y^2 = \lambda \cdot 4y$

$x^2 + 2y^2 = 1$

$\lambda = 1$ or $x = 0$

$3y^2 = y$

$y = \frac{1}{3}, 0$

$2y^2 = 1$

$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$

$\lambda = 0$ unimportant

$x^2 + \frac{2}{9} = 1$

$x = \pm \frac{\sqrt{7}}{3}$

$\left(\pm \frac{\sqrt{7}}{3}, \frac{1}{3}\right)$

$y = 0$

$x = \pm 1$

$(\pm 1, 0)$

$\left(0, \pm \frac{1}{\sqrt{2}}\right)$

contour: $x^2 + 4y^3$

$x^2 + 2y^2 = 1$

evaluate $f(1,0) = f(-1,0) = 1$

$f\left(\frac{\sqrt{7}}{3}, \frac{1}{3}\right) = f\left(-\frac{\sqrt{7}}{3}, \frac{1}{3}\right) = \frac{7}{9} + \frac{4}{27} = \frac{25}{27}$

$f\left(0, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$ ← max

$f\left(0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$ ← min

For functions of 3 variables

- $F(x,y,z)$ $\quad g(x,y,z) = c$

maximize

$$\vec{\nabla} F(x_0,y_0,z_0) = \lambda \vec{\nabla} g(x_0,y_0,z_0) \rightarrow \partial_x F = \lambda \partial_x g$$

etc.

$$g(x,y,z) = c$$

4 eq's for 4 unknowns $(x_0, y_0, z_0), \lambda$.

$V = xyz \qquad 2x + 2y + z = 108$

$V_x = yz \qquad g_x = 2$

$V_y = xz \qquad g_y = 2$

$V_z = xy \qquad g_z = 1$

$yz = 2\lambda$

$xz = 2\lambda \qquad 2x + 2y + z = 108$

$xy = \lambda$

$yz = 2xy \qquad z = 2x \quad (y \neq 0)$

$xz = 2xy \qquad z = 2y \quad (x \neq 0)$

$3z = 108$

$z = 36$

$x = 18$

$y = 18$