So far, lots of attention on two

partial derivatives

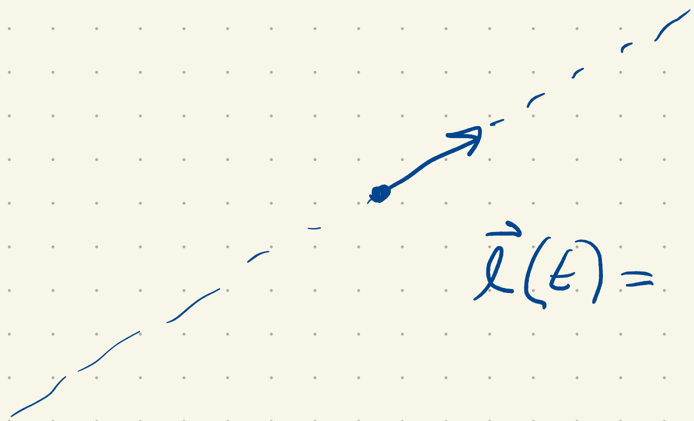
$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$$

These are rates of change in x, y directions

But what about other directions?

$T(x,y) \to$ temperature.

Now our bug walks on a straight line



$$\vec{l}(t) = \langle x_0, y_0 \rangle + \langle a, b \rangle t$$

$T(\vec{l}(t))$ is rate of change of temp as bug moves on a straight line

$$T(\vec{l}(0)) = T(x_0, y_0)$$

$\left.\frac{d}{dt}\right|_{t=0} T(\vec{l}(t))$ is the rate of change of of temp if at (x_0, y_0) and

moves with velocity $\langle a, b \rangle$.

It is known as the directional derivative of T. It needs two ingredients

a) where (x_0, y_0)

b) what velocity $\underbrace{\langle a, b \rangle}_{\vec{v}}$

Your text only wants to use unit vectors $\vec{u}$ and uses the notation $D_{\vec{v}} f(x_0, y_0)$

If f is differentiable, the chain rule applies

$$\frac{d}{dt} f(x_0 + ta, y_0 + tb) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0) b$$

Important special cases: $\vec{v} = \langle 1, 0 \rangle$

$$D_{\vec{v}} f(x,y) = \frac{\partial f}{\partial x}(x_0, y_0)$$

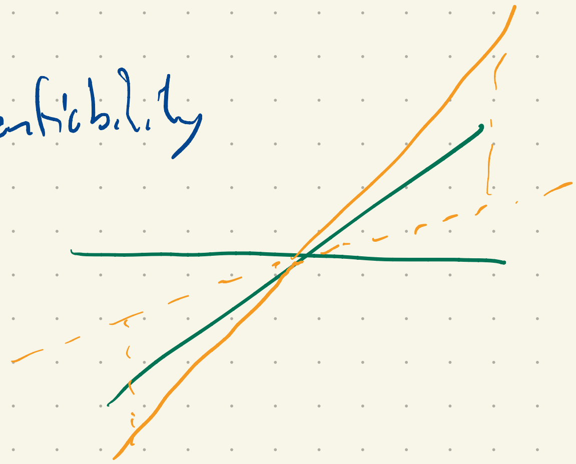
$\vec{v} = \langle 0, 1 \rangle$ $\quad D_{\vec{v}} f(x,y) = \frac{\partial f}{\partial y}(x_0, y_0)$

Upshot: If you know $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$

then you can compute the rate of change of f

in any direction. (!)

This requires differentiability



e.g. $\quad f(x,y) = x\sin(2y)$

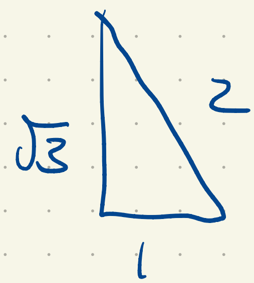
Find directional derivative at $(x,y) = \left(3, \frac{\pi}{3}\right)$

in the direction of a unit vector

with angle $\theta = \pi/6$.

$$\vec{v} = \langle \cos\theta, \sin\theta \rangle =$$

$$= \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$



$$\frac{\partial f}{\partial x} = \sin(2y) \qquad \frac{\partial f}{\partial y} = 2x\cos(2y)$$

$$\frac{\partial f}{\partial x}\left(3, \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) \qquad \frac{\partial f}{\partial y}\left(3, \frac{\pi}{3}\right) = 6\cos\left(\frac{2\pi}{3}\right)$$

$$= 6 \cdot \frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2}$$

$$= 3\sqrt{3}$$

$$D f_{\vec{v}}(x_0, y_0) = -\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} 3\sqrt{3} = \sqrt{3}$$

Now look at

$$\frac{\partial f}{\partial x} \cdot a + \frac{\partial f}{\partial y} \cdot b \qquad \vec{v} = \langle a, b \rangle$$

This looks like a dot product

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \vec{v}$$

we define $\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{\imath} + \frac{\partial f}{\partial y} \hat{\jmath}$

and call it the <u>gradient</u> vector.

It determines a vector at each location

$$\boxed{D_{\vec{v}} f = \vec{\nabla} f \cdot \vec{v}}$$

e.g. $f(x, y) = \frac{1}{2}(x^2 + y^2)$

$$\frac{\partial f}{\partial x} = x \qquad \frac{\partial f}{\partial y} = y$$

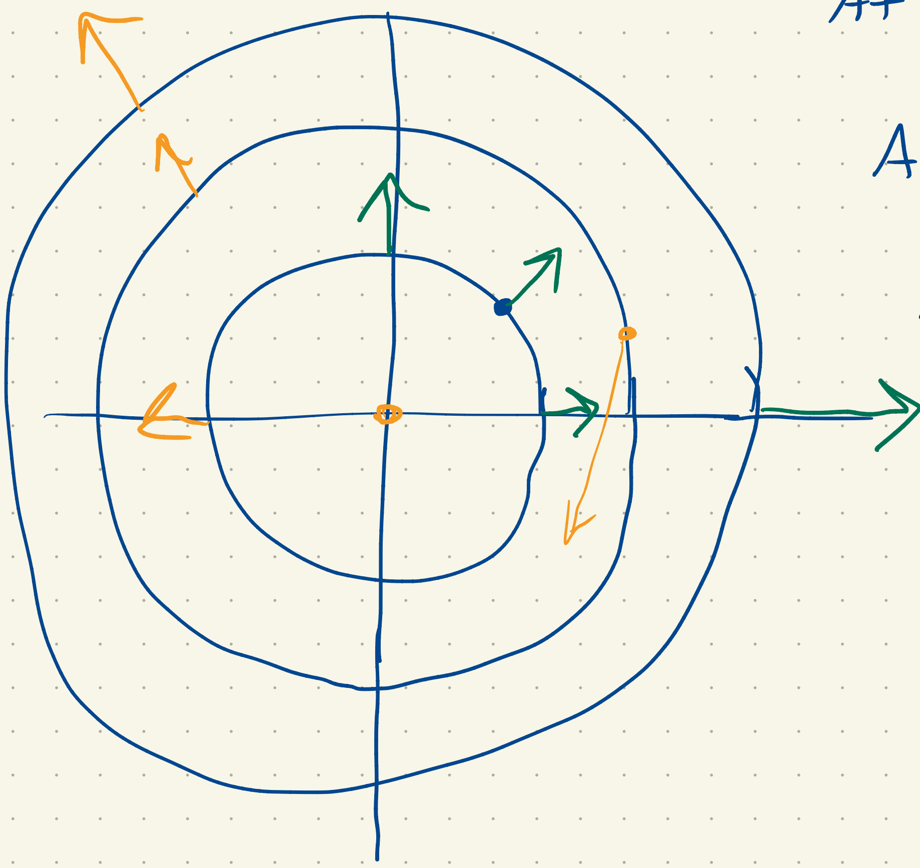
$$\vec{\nabla} f = x\hat{i} + y\hat{j}$$

Compute $D_{\vec{v}} f$ at $(2,1)$ if

$\vec{v} = \langle -1, -3 \rangle$

$$\vec{\nabla} f = \langle 2, 1 \rangle$$

$$\vec{\nabla} f \cdot \vec{v} = -2 - 3 = -5$$



At $(1,1)$, $\vec{\nabla} f = \langle 1, 1 \rangle$

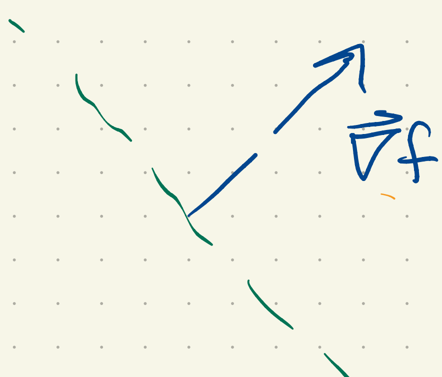
At $(0,1)$ $\vec{\nabla} f = \langle 0, 1 \rangle$

At $(1,0)$ $\vec{\nabla} f = \langle 1, 0 \rangle$

At $(3,0)$ $\vec{\nabla} f = \langle 3, 0 \rangle$

$\vec{\nabla} f \cdot \vec{v}$ tells you how fast f is changing if you move with velocity $\vec{v}$.

Suppose $\vec{\nabla} f \neq 0$ at some point



The directions $\vec{v}$, $\vec{v} \cdot \vec{f} = 0$ form a line