Last class:

Saw the chain rule

$$T(\vec{f}(t)) = T(x(t), y(t))$$

$$\frac{d}{dt} T(\vec{f}(t)) = \underbrace{\frac{\partial T}{\partial x}}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial T}{\partial x}(x(t), y(t)) \frac{dx}{dt} + \cdots$$

What about the other way around?

$$g(T(x,y))$$

Now we need to compute partial derivatives.

e.g. $T(x,y) = x^2+y^2$

$g(z) = e^{-z}$

$$\frac{\partial}{\partial x} g(T(x,y)) = g'(T(x,y)) \cdot \frac{\partial T}{\partial x}$$

(treat y as constant!)

$$-e^{-(x^2+y^2)} \cdot 2x$$

$$\frac{\partial}{\partial x} e^{-(x^2+y^2)} = -e^{-(x^2+y^2)} \cdot (2x)$$

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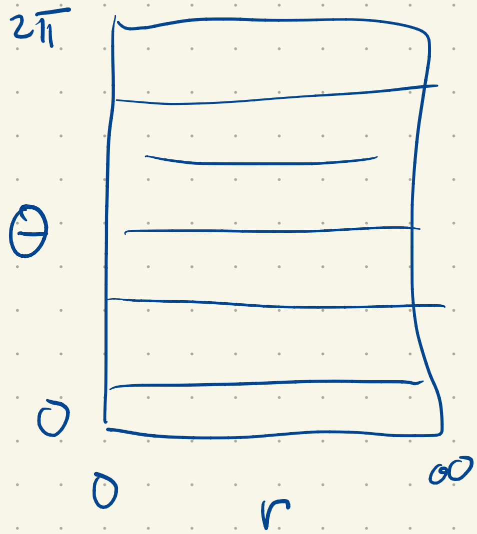
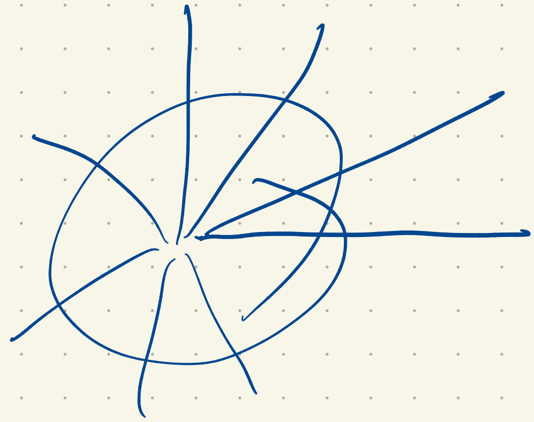
$T(x,y) \qquad x(a,b) \quad y(a,b)$

$$T(x(a,b), y(a,b)) = g(a,b)$$

$$\frac{\partial}{\partial a} T(x(a,b), y(a,b)) = \frac{\partial T}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial a}$$

$$\frac{\partial}{\partial b} T(x(a,b), y(a,b)) = \frac{\partial T}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial b}$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$T(x,y) = x e^{-y} \quad ; \quad \frac{\partial T}{\partial x} = e^{-y} \qquad \frac{\partial T}{\partial y} = -x e^{-y}$$

$$T(x(r,\theta), y(r,\theta)) = \hat{T}(r,\theta)$$

$$\hat{T}(r,\theta) = r\cos\theta\, e^{-r\sin\theta}$$

$$\frac{\partial \hat{T}}{\partial r} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r}$$

$$= e^{-r\sin\theta}\cos\theta - r\cos\theta\, e^{-r\sin\theta}\sin\theta$$

$$= \cos\theta \, e^{-r\sin\theta}\left[1 - r\sin\theta\right]$$

$$\frac{\partial \hat{T}}{\partial r} = \cos\theta \, e^{-r\sin\theta} - r\cos\theta \, e^{-r\sin\theta}\sin\theta$$

$$= \cos\theta \, e^{-r\sin\theta}\left[1 - r\sin\theta\right] \checkmark$$

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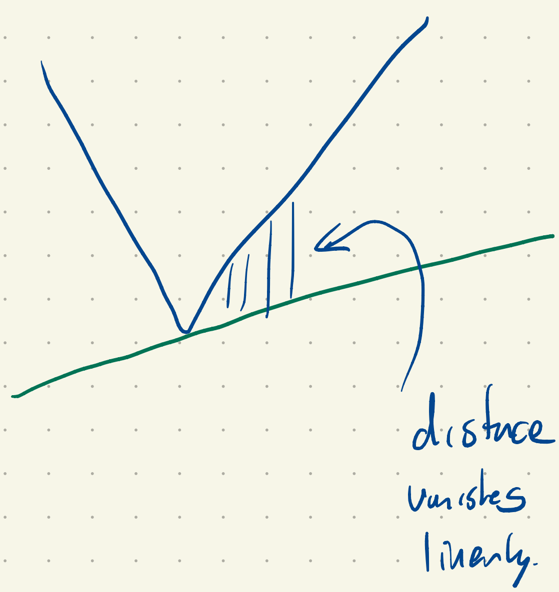
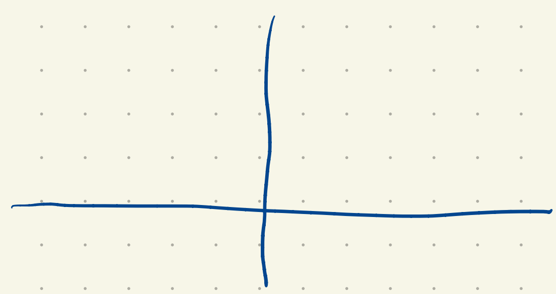
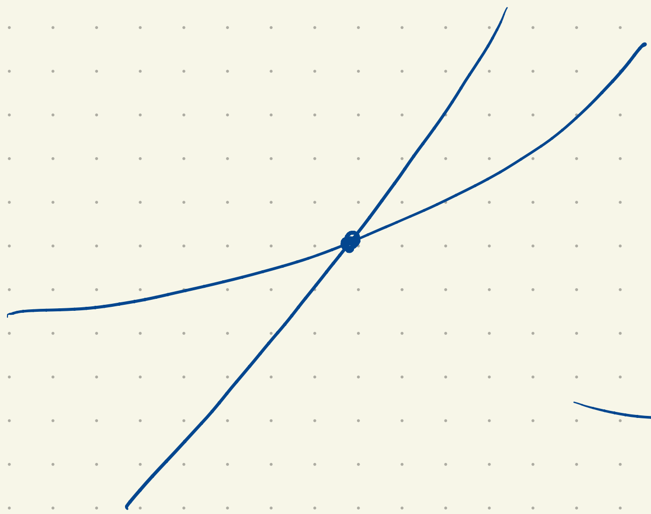
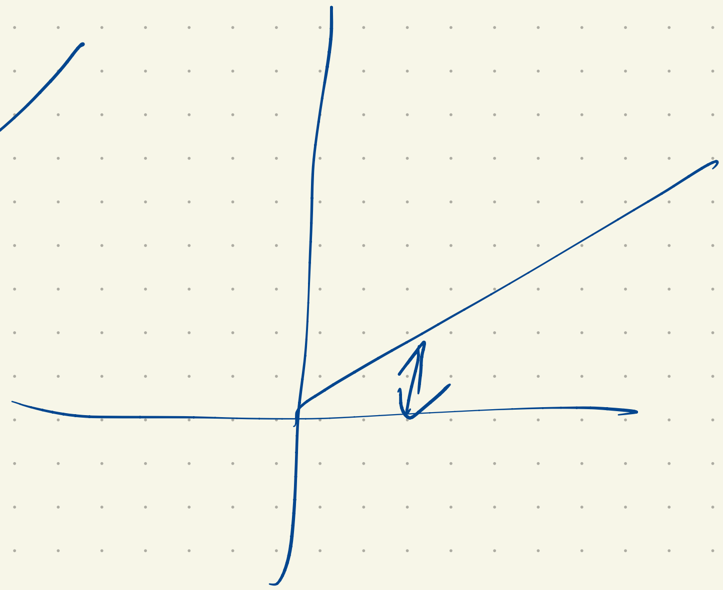
Legalese: the chain rule only works if the function is differentiable.

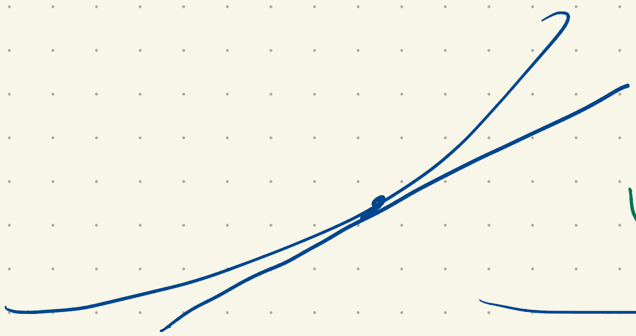
What does this mean?

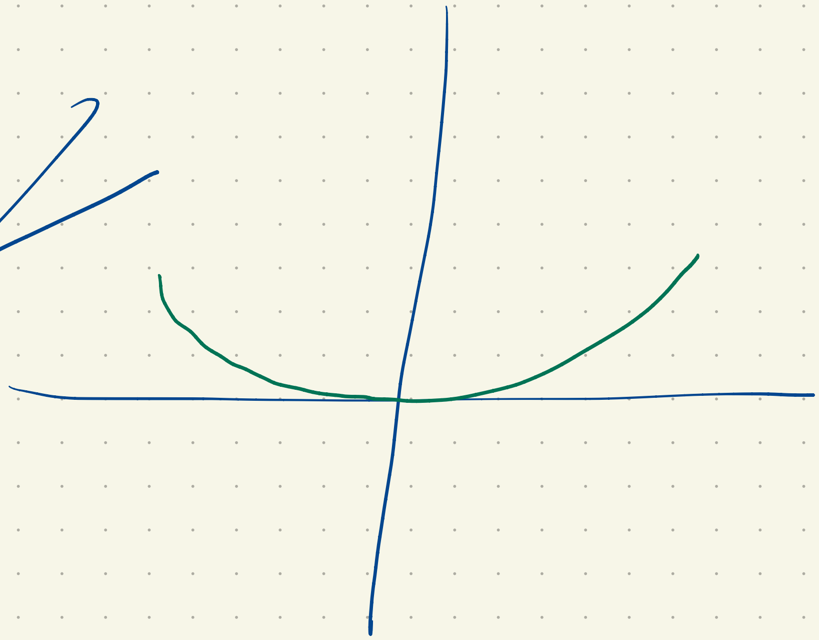










$f(x,y)$

$$\underbrace{f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}_{L(x,y)}$$

Differentiable means

$f(x,y) - L(x,y) \longrightarrow 0$ faster than

$|x-x_0| + |y-y_0|$

(faster than linear).

$f(x,y) = x^2 + 3y^2$

At $(2,1)$ $\qquad L(x,y) = 7 + 4(x-2) + 6(y-1)$

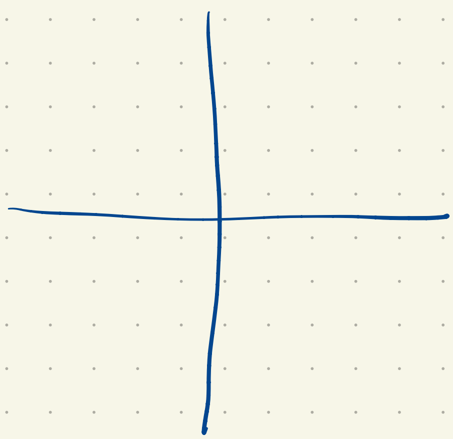
$\frac{\partial f}{\partial x} = 4 \qquad \frac{\partial f}{\partial y} = 6$

Plot graph + linearization

Plot error.

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & x,y \neq 0 \\ 0 & (x,y) = (0,0) \end{cases}$$

$f(0,y) = 0$



$f(x,0) = 0$

$\frac{\partial f}{\partial x}(0,0) = 0$

$\frac{\partial f}{\partial y}(0,0)$

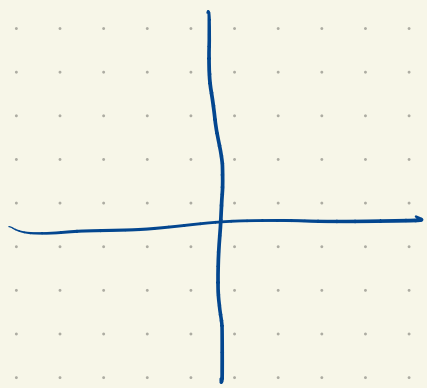
Linearization at $(0,0)$ is $0$.

$L(x,y) = f(0,0) + f_x(0,0)(x-0) + \cdots$

$$= 0 + 0\cdot x + 0\cdot y = 0.$$

$$f(x,x) = \frac{x^3}{2x^2} = \underbrace{\frac{1}{2}x}$$

→ error vanishes linearly.



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If a function has continuous partial derivatives on a disc

then it is differentiable at all points on that disc,

i.e. the error between the func and its linearization vanishes faster than linearly.