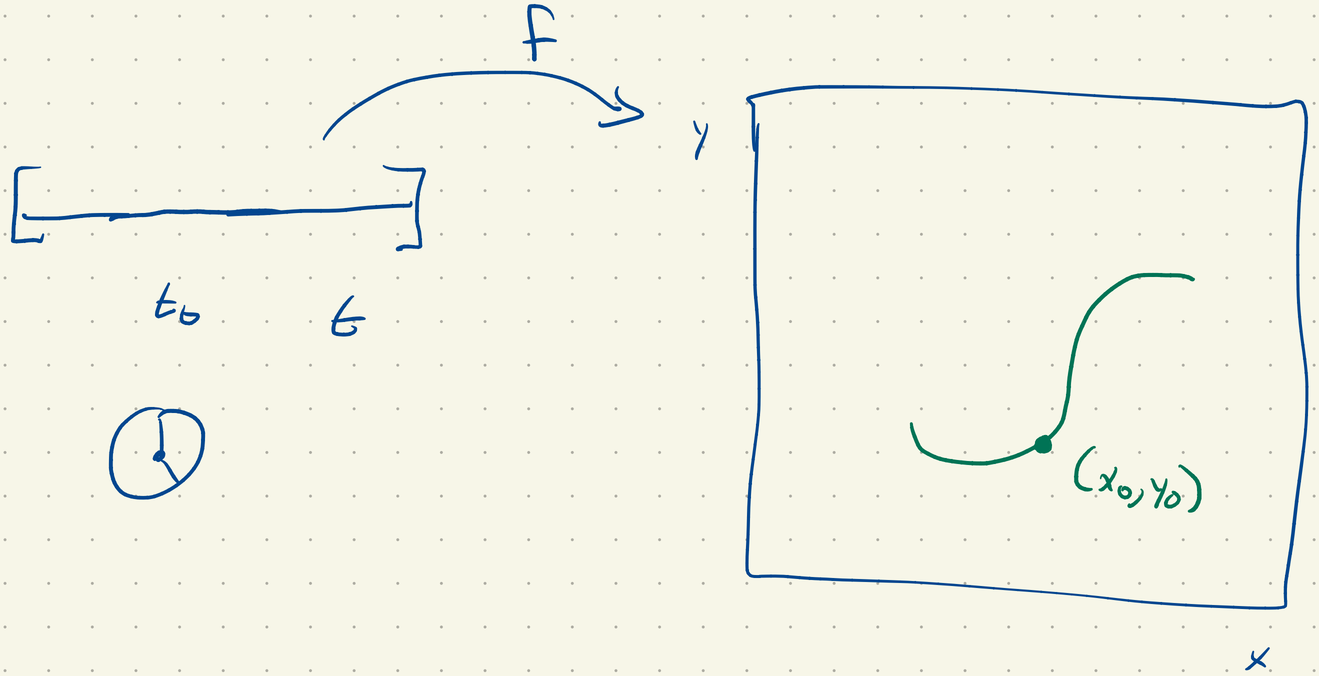
Preamble:

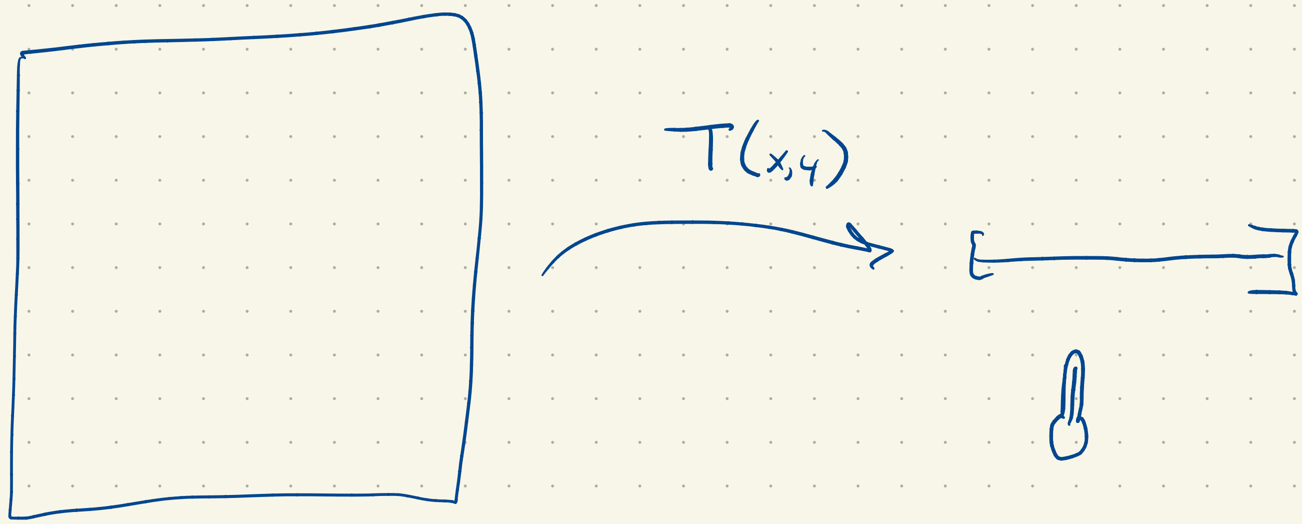
How to compute derivatives of composite functions:

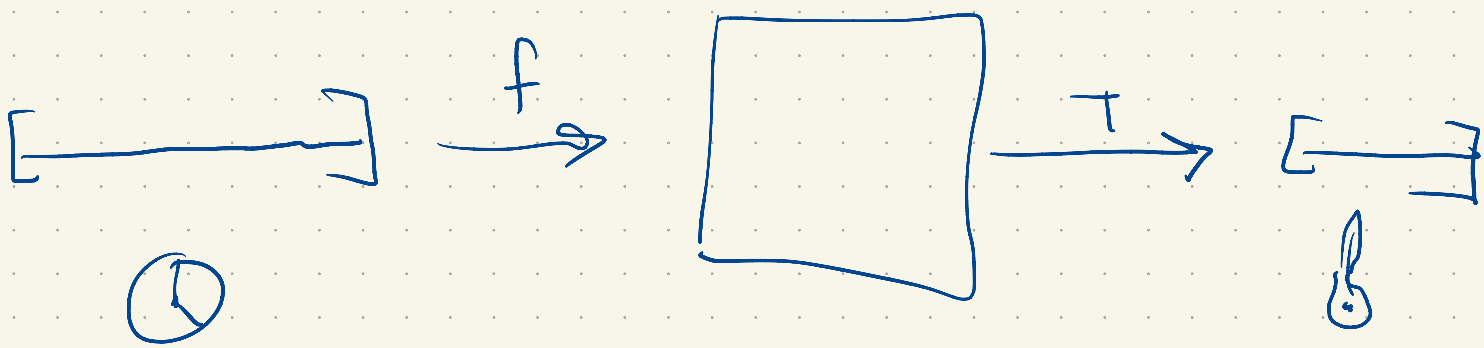
We have a bug. It's position is known at time t



$$f(t) = \langle x(t), y(t) \rangle = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

In the plane there is a temperature

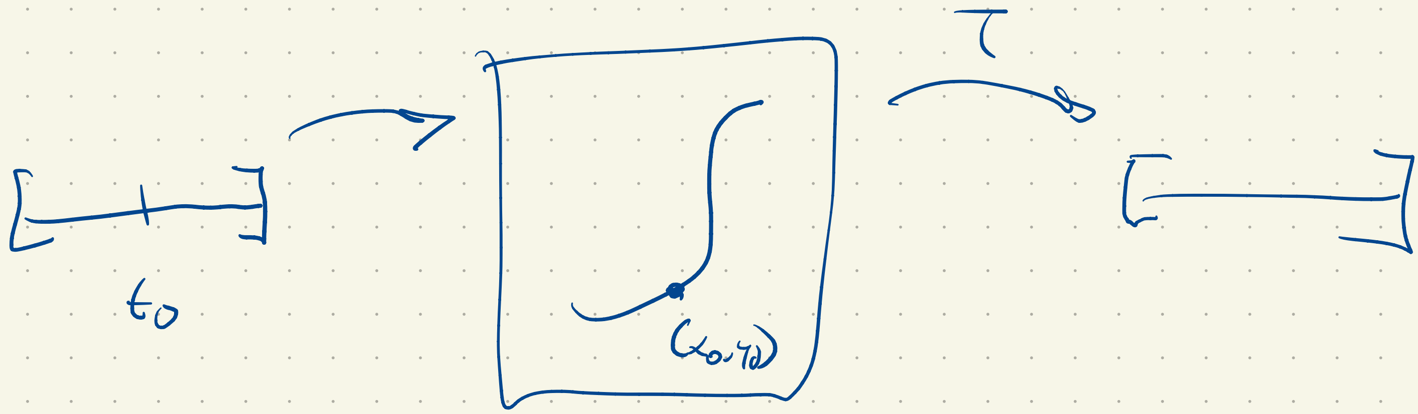




$(T \circ f)(t)$ is the temperature experienced by the bug at time t

$g(t) = (T \circ f)(t)$

$\frac{d}{dt} g(t)$ I'll look at one time t_0



$$f(t_0) = \langle x_0, y_0 \rangle = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

We'll use the linerization

$$g(t) \approx g(t_0) + g'(t_0)(t - t_0)$$

$\uparrow$

going to build this

$$\vec{f}(t) \approx \vec{f}(t_0) + \vec{f}\,'(t_0)(t - t_0) \quad = \langle x_0, y_0 \rangle + \cdots$$

$$= \langle x_0, y_0 \rangle + \langle x'(t_0), y'(t_0) \rangle (t - t_0)$$

$$\langle x - x_0, y - y_0 \rangle \approx \langle x'(t_0)(t - t_0), y'(t_0)(t - t_0) \rangle$$

$$T(x, y) \approx T(x_0, y_0) + \frac{\partial T}{\partial x}(x - x_0) + \frac{\partial T}{\partial y}(y - y_0)$$

$$T(\vec{f}(t)) \approx T(x_0, y_0) + \frac{\partial T}{\partial x}(x_0, y_0)\, x'(t_0)^{(t-t_0)} + \frac{\partial T}{\partial y}(x_0, y_0) \quad c$$

$$g'(t_0) = \frac{\partial T}{\partial x}(x_0, y_0)\frac{dx}{dt}(t_0) + \frac{\partial T}{\partial y}(x_0, y_0)\frac{dy}{dt}(t_0)$$

This is the chain rule.

e.g.

$$T(x, y) = x^2 e^{-y}$$

$$f(t) = \langle t, t^2 \rangle$$

$$g = T \circ f = t^2 e^{-t^2} \quad \text{at } t = 2$$

$$g'(t) = 2t\, e^{-t^2} - 2t^3 e^{-t^2}$$

$$g'(2) = (4 - 16)e^{-4} = -12e^{-4}$$

$$\begin{bmatrix} \frac{\partial T}{\partial x}(x_0, y_0) & \frac{\partial T}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} x'(t_0) \\ y'(t_0) \end{bmatrix}$$

$$\frac{\partial T}{\partial x} = 2xe^{-y} \qquad \frac{\partial T}{\partial y} = -x^2e^{-y}$$

$$x(t) = t \qquad y(t) = t^2$$

$$x'(t) = 1 \qquad y'(t) = 2t$$

$$t_0 = 2 \qquad x_0 = 2 \qquad y_0 = 4$$

$$\frac{\partial T}{\partial x}(x_0, y_0) = 4e^{-4} \qquad \frac{\partial T}{\partial y}(x_0, y_0) = -4e^{-4}$$

$$\frac{dx}{dt}(t_0) = 1 \qquad \frac{dy}{dt}(t_0) = 4$$

$$\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = 4e^{-4} \cdot 1 - 4e^{-4} \cdot 4$$

$$= -12e^{-4} \checkmark$$

e.g. $P = 8.31 \frac{T}{V}$

$T = 300 \quad \frac{dT}{dt} = 0.1 K/s$

$V = 100 L \quad \frac{dV}{dt} = 0.2 L/s$

$$\frac{dP}{dt} = \left[\frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} \right]$$

$$= 8.31 \left[\frac{1}{V} \frac{dT}{dt} - \frac{T}{V^2} \frac{dV}{dt} \right]$$

$$= 8.31 \left[\frac{1}{100} \frac{1}{10} - \frac{300}{100^2} \frac{2}{10} \right]$$

$$= 8.31 \frac{1}{100} \cdot \frac{1}{10} \left[1 - 6 \right]$$

$$= - \frac{8.31}{100} \cdot \frac{1}{2} = - 0.04155 \text{ Pa/s}$$