8) compositions of $f(g(x,y))$

Last class

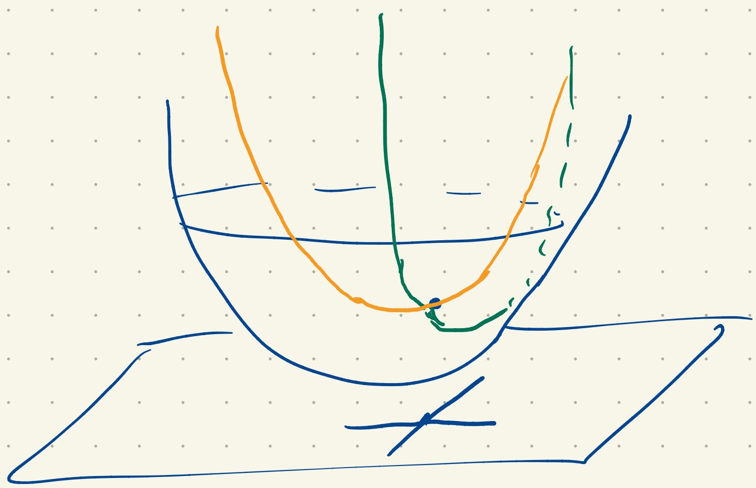
$f(x,y)$

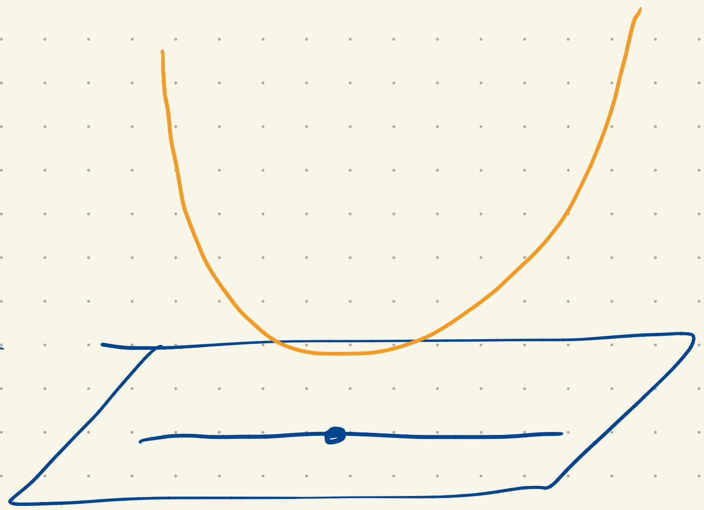
$\frac{\partial f}{\partial x}$: treat y as constant

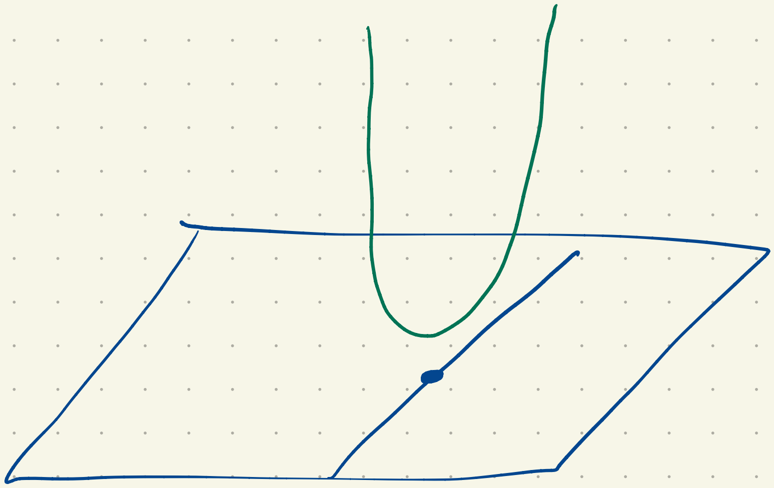
$$f(x,y) = \sin(x^2 y)$$

$$\frac{\partial f}{\partial x} = \cos(x^2 y)\, 2xy$$

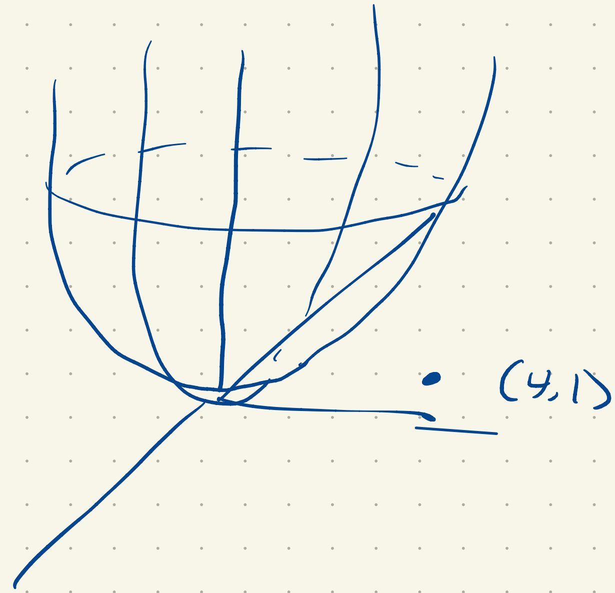
$$\frac{\partial f}{\partial y} = \cos(x^2 y)\, x^2$$







$f(x,y) = x^2 + 3y^2$



$(a,b) = (4,1)$

$f_x(2,1) = 2 \cdot 4 = 8$

$f_y(2,1) = 6$

f is increasing more steeply in the x - direction

14.3 (continued)

2^{nd} partial derivatives

$$f(x,y) = \sin(x^2 y)$$

$$\frac{\partial f}{\partial x} = \cos(x^2 y) \cdot 2xy$$

$$\frac{\partial f}{\partial y} = \cos(x^2 y)\, x^2$$

How does f change in x, y directions

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x} = -\sin(x^2 y)\, 2x^3 y + \cos(x^2 y)\, 2x$$

How does $\frac{\partial f}{\partial x}$ change in y direction?

$$= 2x\left[-x^2 y \sin(x^2 y) + \cos(x^2 y)\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -\sin(x^2 y)\, 2x^3 y + 2x\cos(x^2 y)$$

$$= 2x\left[\cos(x^2 y) - x^2 y \sin(x^2 y)\right]$$

Remarkable: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ in this case.

$$\partial_x \left[\frac{x^3 y - y^3 x}{x^2 + y^2} \right] = \frac{[3x^2 y - y^3](x^2+y^2) - [x^3 y - y^2 x]\, 2x}{(x^2+y^2)^2}$$

at $x=0$: $\frac{-y^5}{y^4} = -y$

$\partial_y \partial_x f = -1$ on line $x = 0$

$$\partial_y f = \frac{(x^3 - 3y^2 x)(x^2+y^2) - (x^3 y - y^3 x)(2y)}{(x^2+y^2)^2}$$

at $y=0$:

$$\frac{x^3 \cdot x^2}{x^4} = x$$

$\partial_x \partial_y f = 1$

$\partial_x \partial_y f \neq \partial_y \partial_x f$

on line $y = 0$

Thm: If f_{xy} and f_{yx}

exist on a disk containing (a,b)

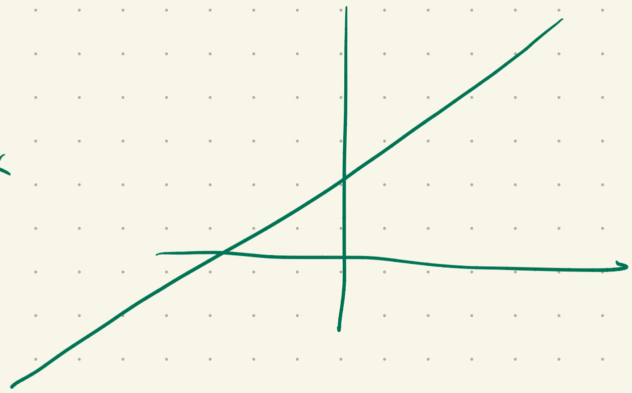
and are continuous, then

$$f_{xy} = f_{yx}.$$

(2^{nd} partials agree)

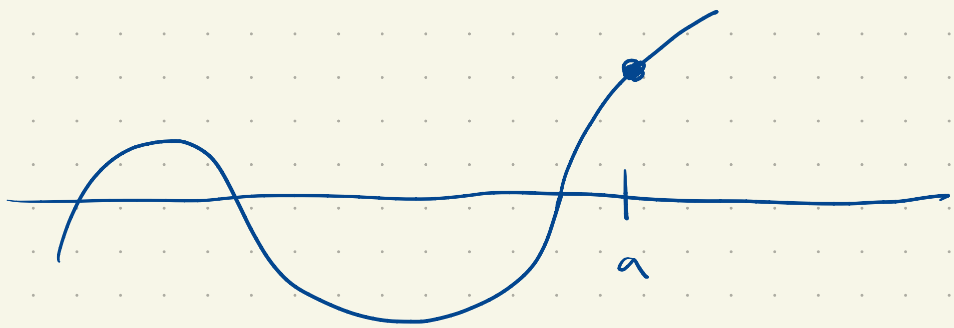
Linear approximation.

$f(x) = 5 + 7x$



In some sense, these are the next most complicated functions, after the constants.

Recall from calc I



Linearization of $f(x)$ at $x=a$

$L(x) = A + B(x-a)$ A, B numbers,

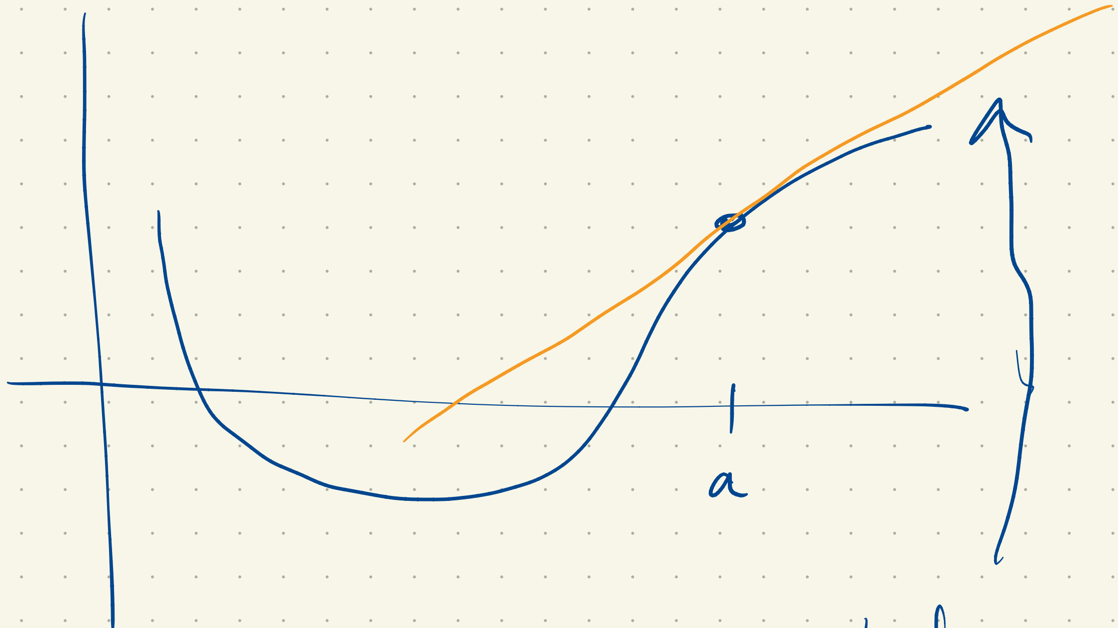
best approximates $f(x)$ "near" $x=a$.

How good?

$L(a) = f(a) \Rightarrow A = f(a)$

$L'(a) = f'(a) \Rightarrow B = f'(a)$

$L(x) = f(a) + f'(a)(x-a)$



graph of

linearization.

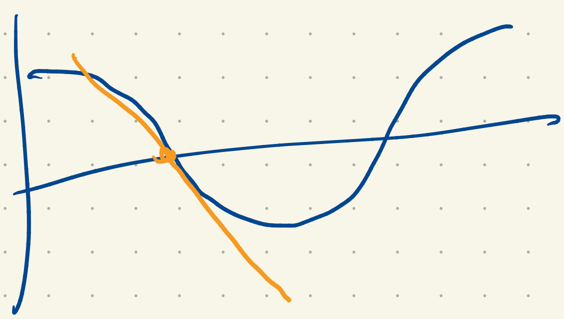
e.g. $f(x) = \cos(x)$

$a = \frac{\pi}{2}$

$f(\pi/2) = 0$

$f'(\pi/2) = -1$

$L(x) = -(x - \pi/2) = \pi/2 - x$



Generalization to two input variables

$$f(x,y) = 3 + 2x - y$$

number

·umber   number

$Ax + By + C \qquad A, B, C \in \mathbb{R}$

$C + A(x-a) + B(y-b) \longrightarrow (C - aA - bB + Ax + By)$

What does the graph of such a function look like?

$z = 3 + 2x - y$

$-2x + y + z = 3$ aha! It's graph is a plane.

Functions of the form are called affine (loosely linear)

The linearization of a function $f(x,y)$ at (a,b)

a function $L(x,y)$ of the form $C + A(x-a) + B(y-b)$

that "best approximates" $f(x,y)$ near (a,b).

How good?

$$L(a,b) = f(a,b) \Rightarrow C = f(a,b)$$

$$\frac{\partial L}{\partial x}(a,b) = f_x(a,b) \Rightarrow A = f_x(a,b)$$

$$\frac{\partial L}{\partial y}(a,b) = f_y(a,b) \Rightarrow B = f_y(a,b)$$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

e.g. Compute the linearization of

$$f(x,y) = x^2 + 3y^2 \quad \text{at} \quad (a,b) = (2,1)$$

$$f(2,1) = 4 + 3 = 7$$

$f_x(x,y) = 2x$	$f_y(x,y) = 6y$
$f_x(2,1) = 4$	$f_y(2,1) = 6$

$$L(x,y) = 7 + 4(x-2) + 6(y-1)$$

The graph of the linearization is known as the tangent plane (at a,b).

[MATLAB]
