14.2 Limits, Continuity

Why do we care about limits? $\frac{0}{0}$ is the main suspect.

average rate of change $\frac{x(t+h)-x(t)}{h}$

but put $h=0$ $\frac{x(t)-x(t)}{0}$ $\frac{0}{0}$ oops!

We can still ask what happens as $h \to 0$

$\frac{\sin(x)}{x}$ not defined at $x=0$

But $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ [Try it!]

We need limits to define derivatives of multivariable functions

so we might as well talk about

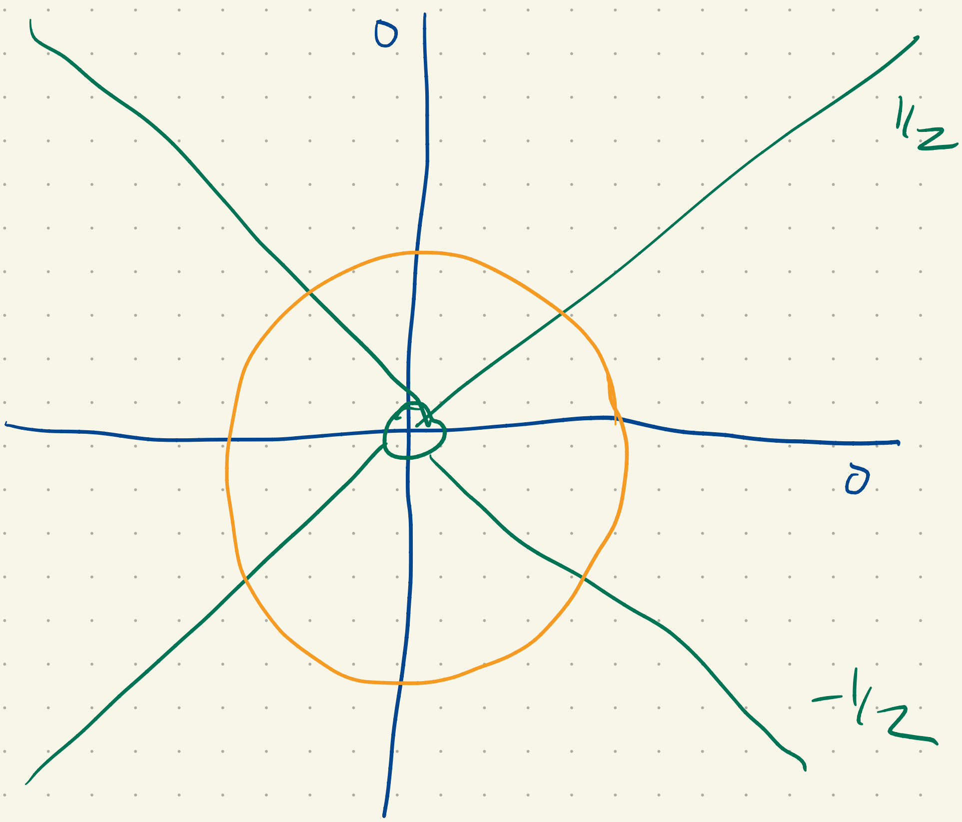
them now.

$f(x,y) = \dfrac{xy}{x^2+y^2}$ is a fun function.

$x=0, y=0$ $\quad \frac{0}{0}$, uh oh. not defined at $(0,0)$

We can still ask whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists.

Contour plot:

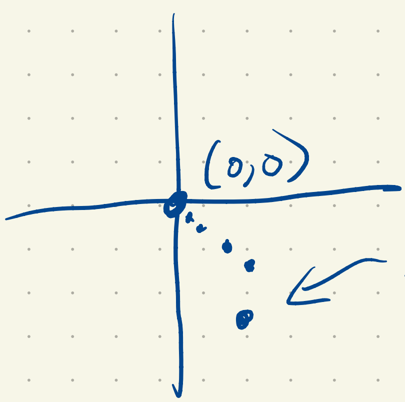


$y = x \qquad \dfrac{x^2}{x^2+y^2} = \dfrac{1}{2}$

$y = -x \qquad \dfrac{-x^2}{x^2+y^2} = -\dfrac{1}{2}$

$x = \cos\theta \quad y = \sin\theta \qquad \dfrac{\sin\theta\cos\theta}{1} = \dfrac{1}{2}\sin 2\theta$

$x = r\cos\theta \quad y = r\sin\theta$ same! (does not depend on r)



sequence of inputs (x_n, y_n)

$x_n \to 0$

$y_n \to 0$

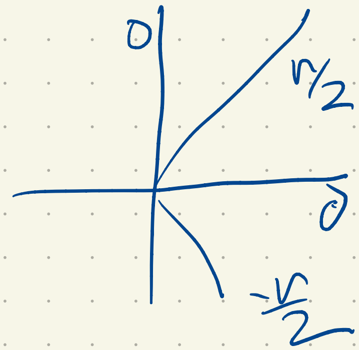
$z_n = f(x_n, y_n)$

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$

needs $z_n \to L$

no matter what sequence you pick.

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \qquad \frac{r\cos\theta\, r\sin\theta}{r} = r\sin(2\theta)$$



Still 0/0

$x_n \to 0 \qquad x_n^2 \to 0$ also

$y_n \to 0 \qquad y_n^2 \to 0$ also

$r_n = \sqrt{x_n^2+y_n^2} \to 0$ (continuity)

$$-r_n \le f(x_n,y_n) \le r_n$$

Squeeze theorem $\quad r_n \to 0$, $-r_n \to 0 \Rightarrow f(x_n,y_n) \to 0.$

Plot this in Matlab.

To show $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist, one

approach: find two sequences $x_n \to a$, $y_n \to b$ $\quad \hat{x}_n \to a$, $\hat{y}_n \to b$

such that $z_n = f(x_n, y_n)$

$$\hat{z}_n = f(\hat{x}_n, \hat{y}_n)$$

$$z_n \to L_1 \qquad \hat{z}_n \to L_2 \qquad L_1 \neq L_2$$

e.g. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$

$$f(x, mx) = \frac{x m^2 x^2}{x^2 + m^4 x^4} = \frac{x m^2}{1 + m^4 x^2} \longrightarrow 0$$

as $x \to 0$

$f\left(\frac{1}{n}, 0\right) = 0$ for all n.

$x = y^2 \qquad \dfrac{y^4}{y^4 + y^4} = \dfrac{1}{2} \quad (y \neq 0)$

$(x_n, y_n) = \left(\dfrac{1}{n^2}, \dfrac{1}{n} \right) \qquad f(x_n, y_n) = \dfrac{1}{2}$

Continuity:

We say $f(x,y)$ is cts at (a,b) if

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b).$$

cts $\Rightarrow$ continuous on all domain

It's a question of approximation.

$(x_n, y_n) \to (a,b)$

$f(x_n, y_n) \to f(a,b)$

error on inputs small $\Rightarrow$ error in output small.

Continuous functions: (of x, y)

1) constants,

2) x

3) y

4) sums, products, differences of cts functions

$$f(x,y) = xy$$

$$f(x,y) = 1 + xy$$

$$f(x,y) = 1 + 7xy$$

4') polynomials in x, y

5) old friends: sin, cos, ln, exp, arctan

on their domains

6) quotients $\dfrac{f(x,y)}{g(x,y)}$ on domain $(g(x,y) \neq 0!)$

7) rational functions $\dfrac{p(x,y)}{q(x,y)}$

8) compositions of $f(g(x$