$\vec{a} \times \vec{b} = 0$ means what?

$$|\vec{a}|\,|\vec{b}| \sin\theta = 0$$

$\vec{a} = 0$ or $\vec{b} = 0$ or

$\sin\theta = 0$ $\quad 0 \le \theta \le \pi$

$\Rightarrow \theta = 0, \pi$

$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are colinear.

---

Algebra rules:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$$

But:

$$\left.\begin{aligned} \vec{i}\times(\vec{i}\times\vec{j}) &= \vec{i}\times\vec{k} = -\vec{j} \\ (\vec{i}\times\vec{i})\times\vec{j} &= \vec{0}\times\vec{j} = \vec{0} \end{aligned}\right] \text{ not assoc!}$$

Instead:

$$\vec{a}\times(\vec{b}\times\vec{c}) = \alpha\,\vec{b} + \beta\,\vec{c}$$

$$= (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$$

$$\vec{i}\times(\hat{i}\times\hat{j}) = \hat{i}\times k = -\hat{j}$$

$$(\hat{i}\cdot\hat{j})\,\hat{i} - (\hat{i}\cdot\hat{i})\,\hat{j} = -\hat{j} \checkmark$$

Triple Product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

Just a computation.

$$= - \begin{vmatrix} \vec{a} \\ \vec{c} \\ \vec{b} \end{vmatrix}$$

$$= + \begin{vmatrix} \vec{c} \\ \vec{a} \\ \vec{b} \end{vmatrix}$$

$$= \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c}$$

If you believed me about volume of parallelopiped, up to sign, this gives geometric meaning to the triple product.

# 12.5 Equations of lines, planes

$$y - y_0 = m(x - x_0)$$

Lines in plane: $y = mx + b$

Most lines have this relation.

$x = x_0$ also!

This is ok if $h(t) = 5t + 9$

(each $t$ has a unique $h$)

But not quite sufficient for geometry.

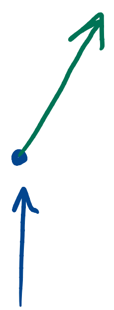
We can specify a line more flexibly, but less uniquely, by giving

a) a point on the line

b) a non zero vector parallel to the line.

1) Blur the distinction between points + vectors: identify each point with displacement from origin.

2)



$\vec{v} = \langle v_1, v_2 \rangle$

$\vec{r}_0 = \langle x_0, y_0 \rangle$

$$\vec{r} = \vec{r}_0 + t\,\vec{v} \qquad t \in \mathbb{R}$$

$$= \langle x_0 + t v_1,\ y_0 + t v_2 \rangle$$

What's the slope?

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$

$$\vec{r}(1) = \langle x_0 + v_1,\ y_0 + v_2 \rangle$$

Rise: $y_0 + v_2 - y_0 = v_2$

run: $x_0 + v_1 - x_0 = v_1$

$$m = v_2 / v_1$$

$$\vec{w} = 7\vec{v} = \langle 7v_1, 7v_2 \rangle$$

$\vec{r} = \vec{r_0} + t\vec{w}$ describes same line

$$m = \frac{7v_2}{7v_1} = v_2/v_1$$

☺

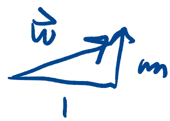
Point: $\langle x_0, y_0 \rangle$ as before. Fantastic.

But we can describe "vertical" lines this way: $\vec{v} = \langle 0, 1 \rangle$

$\vec{v} = \langle 0, -19 \rangle$, etc.

$$\vec{v} = \langle v_1, v_2 \rangle$$

$$\vec{w} = \langle 1, v_2/v_1 \rangle = \langle 1, m \rangle$$



$$\cdot\ \vec{r} = \langle x_0, y_0 \rangle + \langle 1, m \rangle t$$

$$= \langle x_0 + t,\ mt + y_0 \rangle$$

↳ $x$ is $t$!

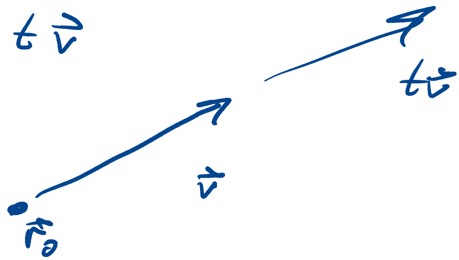
$$x = x_0 + t$$

$$y = mt + y_0$$

$$y - y_0 = m(x - x_0)$$

Same technique works in every dimension.

$$\vec{r} = \vec{r}_0 + t\vec{v}$$




$$\vec{r} = \langle x, y, z \rangle$$

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\vec{v} = \langle a, b, c \rangle$$

$$\boxed{\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}}$$

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"Parametric equations of line"

vs "vector equation"

How many parameters to specify a line?

$\mathbb{R}^2$: 2 parameters (one for angle via $\tan\theta$

one for which line with that slope

$\mathbb{R}^3$: 4 parameters (two for direction

two for which line

in that direction)

$x_0 \quad y_0 \quad z_0$

$\longrightarrow$ redundant info

$a \quad b \quad c$

$\vec{r}_0\,' = \vec{r}_0 + t_0 \vec{v}$ works for any $t_0$

$\vec{v}\,' = \lambda \vec{v}$ works for any $\lambda \neq 0$

Symmetric form $\dfrac{x-x_0}{a} = t$

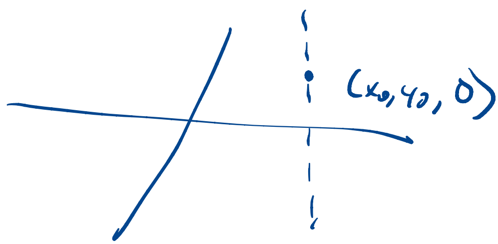
$$\frac{y-y_0}{b} = t$$

$$\frac{z-z_0}{c} = t$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad \text{if } a, b, c \neq 0.$$

If $a = 0$ $\quad x = x_0 \quad \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

If $a=0$, $b=0$ $\quad x = x_0 \quad y = y_0 \quad z$ arbitrary



E.g. Find parametric & symmetric equations of line through $(2,-4,1)$ and $(8,-3,-1)$

point: $(2,-4,1)$

vector $\langle 6, 1, -2\rangle$

$$\vec{r} = \langle 2,-4,1\rangle + \langle 6t, t, -2t\rangle$$

$$x = 2 + 6t$$

$$y = -4 + t$$

$$z = 1 - 2t$$

$$\frac{x-2}{6} = y+4 = \frac{z-1}{-2}$$

what if $v = \langle 12, 2, -4\rangle$

$$\frac{x-2}{12} = \frac{y+4}{2} = \frac{z-1}{-4}$$ mult by 2!

Where does this line intersect $xy$ plane?

$$z = 0 \Rightarrow t = \frac{1}{2}$$

$$x = 2 + 3 = 5$$

$$y = -4 + \frac{1}{2} = -\frac{7}{2}$$

$$(5, -7/2, 0)$$

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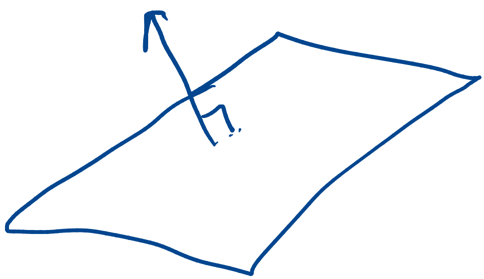
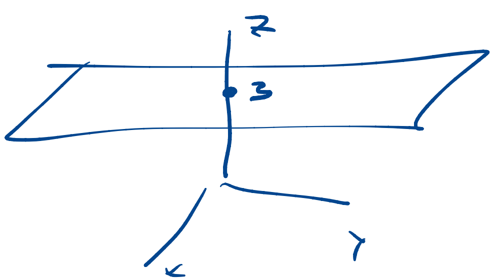
How to describe planes:



In 3-d, every plane has a unique orthogonal direction.

We call a vector ortho to plane a normal vector

$z = 3$




normal vector:

$$\langle 0, 0, 1 \rangle$$