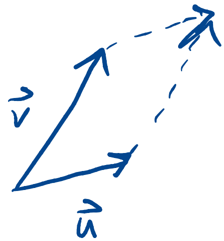
Last class:

$$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 V_1$$



Is the area of this parallelogram, up to sign.

$> 0 \Rightarrow$ turn left from $\vec{u}$ to $\vec{v}$

$< 0 \Rightarrow$ turn right

$= 0 \Rightarrow$? $\vec{u}$ and $\vec{v}$ are colinear.

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ V_1 & v_2 & V_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} V_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}$$

$$+ u_3 \begin{vmatrix} v_1 & V_2 \\ w_1 & w_2 \end{vmatrix}$$

Equals, up to sign, volume of:

positive if curl fingers (right hand) $\vec{u}$ to $\vec{v}$

has thumb on same side as $\vec{w}$.

Introduced $\vec{a} \times \vec{b}$, a vector.

$$\vec{a} \times \vec{a} = 0 \quad \forall \vec{a}$$

$$\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$$

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$$

($\vec{a} \times \vec{b}$ is perp to $\vec{a}, \vec{b}$)

$\vec{a}$ is perpendicular to $\vec{a} \times \vec{b}$.

geometry!

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = -\vec{b} \cdot (\vec{b} \times \vec{a}) = -0 = 0.$$

$\vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ also.

Key Property: $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Mnemonic for computing, using determinants

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$(a_2 b_3 - a_3 b_2)\,\vec{i}$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (a_3 b_1 - a_1 b_3)\,\vec{j}$$

$$- \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j}$$

(Note swapped order!)

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (a_1 b_2 - a_2 b_1)\,\vec{k}$$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Cross product is sum of all 3.

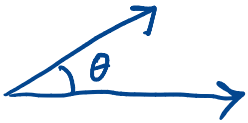
$$\vec{i} \times \vec{j} \qquad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + 1\,\vec{k} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}$$

$$\vec{k} \times \vec{i} = \vec{j}$$

$i \quad j \quad k \quad i \quad j$

Recall $\qquad \vec{a} \cdot \vec{b} = |a|\,|b| \cos\theta$



$0 < \theta < \pi$

$$\vec{a} = |\vec{a}|\,\vec{i}$$

$$\vec{b} = |\vec{b}| \cos\theta\,\vec{i} + |\vec{b}| \sin\theta\,\vec{j}$$

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\,\vec{i} \times \vec{j}$$

$$= |\vec{a}|\,|\vec{b}| \sin\theta\,\vec{k}$$

$$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta}$$

> 0

This is always true.

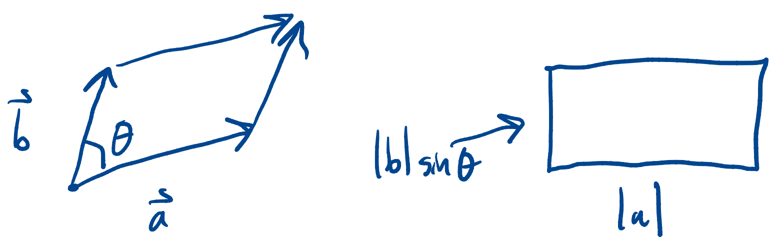
$\vec{u} = \langle u_1, u_2, 0\rangle$

$\vec{v} = \langle v_1, v_2, 0\rangle$

$$\vec{u}\times\vec{v} = (u_1 v_2 - v_2 u_1)\,\vec{k}$$

↳ this is a 2×2 determinant

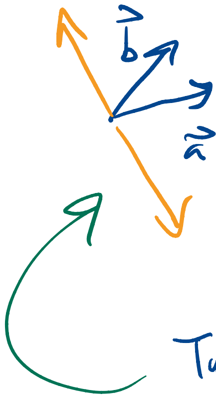
$|\vec{u}\times\vec{v}|$ is area of parallelogram spanned by $\vec{u}, \vec{v}$.



$|\vec{a}\times\vec{b}|$ is the area of the parallelogram spanned by $\vec{a}, \vec{b}$.

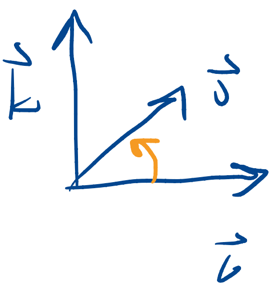
So $\vec{a}\times\vec{b}$ is perp to $\vec{a}$, and perp to $\vec{b}$.

We know its length. Great.



Two possibilities.

$\vec{i}\times\vec{j}=\vec{k}$



Right hand fingers curl from $\vec{a}$ to $\vec{b}$, thumb is on same side as $\vec{a}\times\vec{b}$.

$\vec{a} \times \vec{b} = 0$ means what?

$|\vec{a}|\,|\vec{b}| \sin\theta = 0$

$\vec{a} = 0$ or $\vec{b} = 0$ or

$\sin\theta = 0 \qquad 0 \le \theta \le \pi$

$\Rightarrow \theta = 0, \pi$

$\vec{a} \times \vec{b} = 0 \Leftrightarrow$ $\vec{a}$ and $\vec{b}$ are colinear.

Algebra rules:

$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

But:

$$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$$
$$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$$

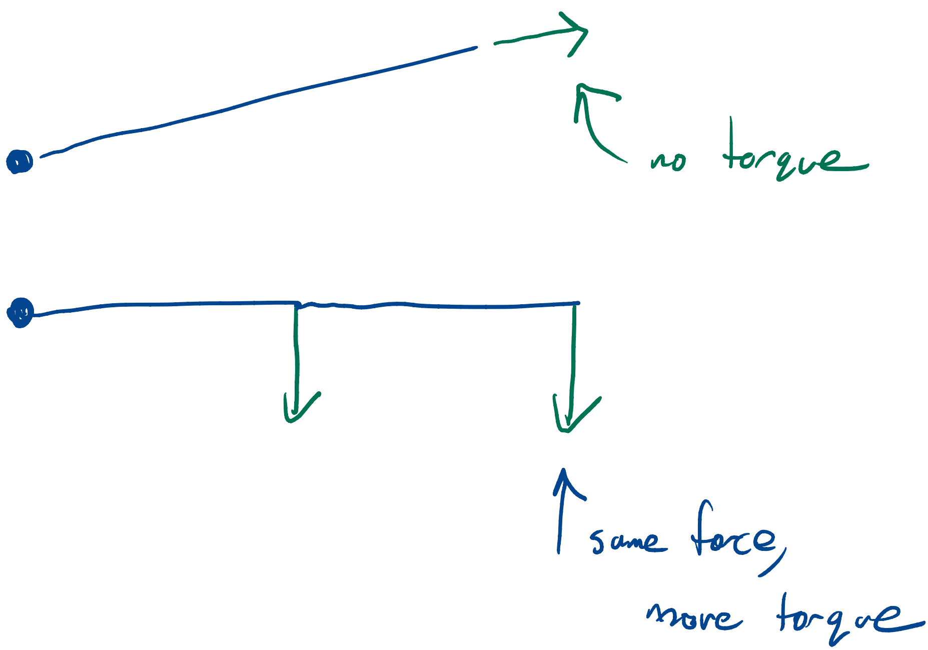
not assoc!

Instead:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \alpha \vec{b} + \beta \vec{c}$$
$$= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

$$\vec{i} \times (i \times j) = i \times k = -j$$
$$(i \cdot j) i - (i \cdot i) j = -j \checkmark$$

Torque: (A vector!)



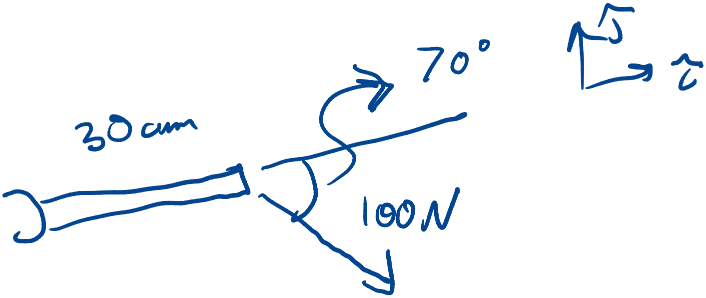
By definition, the torque at P of $\vec{F}$

is $\vec{r} \times \vec{F} = \vec{\tau}$ (Units: Nm, newton metres)

If $\vec{F}$ is parallel to $\vec{r}$ $\vec{r} \times \vec{F} = \vec{0}$

$$|\vec{r} \times \vec{F}| = |\vec{r}|\,|\vec{F}| \sin\theta$$

maximised when $\theta = \frac{\pi}{2}$



Determine the size and direction of the torque vector:

$$\vec{F} = 100\,N\cos(70^\circ)\,\hat{i} - 100\,N\sin(70^\circ)\,\hat{j}$$

$$0.3\,m\,\hat{i} \times \vec{F} =$$

$\sin(70^\circ)$

$$-0.3 \cdot 100\,Nm\,\hat{i} \times \hat{j}$$

$$-28.19\,Nm\,\hat{k}$$

28.2 Nm into page

Triple Product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix}$$

Just a computation.

$$= - \begin{vmatrix} \vec{a} \\ \vec{c} \\ \vec{b} \end{vmatrix}$$

$$= + \begin{vmatrix} \vec{c} \\ \vec{a} \\ \vec{b} \end{vmatrix}$$

$$= \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$= (\vec{a} \times \vec{b}) \cdot \vec{c}$$

If you believed me about volume of parallelopiped, up to sign, this gives geometric meaning to the triple product.