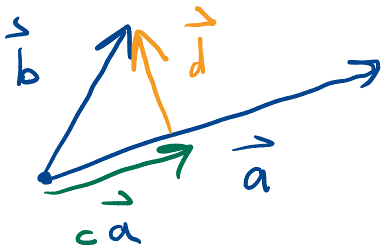
Orthogonal Projection

The dot product measures, in some sense, how alike two vectors are.

An [illegible] cons...



I want to write $\vec{b}$ as a sum of two pieces. One is in the direction of $\vec{a}$.

The other is orthogonal to $\vec{a}$.

$$\vec{b} = c\vec{a} + \vec{d} \qquad \vec{} \cdot \vec{d} = 0$$

$$\vec{b} \cdot \vec{a} = c\,\vec{a} \cdot \vec{a} + \vec{d} \cdot \vec{a}$$

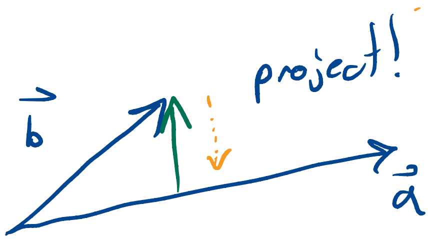
$$= c\,|\vec{a}|^2$$

$$c = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}$$

$$\vec{b} = \underbrace{\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}}\ \vec{a} + \vec{d}$$

The orthogonal projection of $\vec{b}$ onto $\vec{a}$.

$$\mathrm{proj}_{\vec{a}}\, \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\, \vec{a}$$



$$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\, \vec{a} = \vec{b} \cdot \Bigg(\underbrace{\frac{\vec{a}}{|\vec{a}|}}\Bigg)\, \underbrace{\frac{\vec{a}}{|\vec{a}|}}$$

unit vector version of $\vec{a}$

$$\operatorname{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \qquad \left| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \vec{a} \right| = \left| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} \right|$$

It is the signed magnitudes of the orthogonal projection

$$\vec{b} = 5\vec{i} + 2\vec{j} - 6\vec{k}$$

$$\vec{a} = \vec{k}$$

$$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} = \vec{b} \cdot \vec{a}\ \vec{k} = -6k$$

$$\vec{a} = 9\vec{k}$$

$$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} \vec{a} = \frac{\vec{b} \cdot (9\vec{k})(9\vec{k})}{9^2}$$

$$= (\vec{b} \cdot \vec{k})\vec{k} = -6\vec{k} \quad \text{still}$$

$$-b = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \vec{b} \cdot \vec{k} = -6.$$

Section 12.4 Cross Product

Warmup exercise: 2x2 determinant

$\vec{u} = \langle u_1, u_2 \rangle$

$\vec{v} = \langle v_1, v_2 \rangle$

$$\begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix} := \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

2x2 matrix

By definition,

2x2 determinant

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

need not be positive.

1)

$$\begin{vmatrix} u_1 & u_2 \\ u_1 & u_2 \end{vmatrix} = u_1 u_2 - u_2 u_1 = 0$$

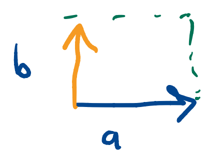
2) $\begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} = - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

$\downarrow$

$v_1 u_2 - v_2 u_1 = -(u_1 v_2 - u_2 v_1)$

3) $\vec{u} = \langle a, 0 \rangle$

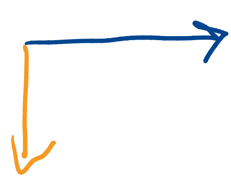
$\vec{v} = \langle 0, b \rangle$



$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$

$\rightarrow$ area of parallelogram

But: if $a > 0$, $b < 0$



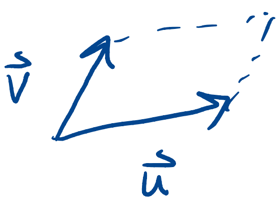
then $|ab|$ is the area of the

parallelogram, but $\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix}$ is negative

Fact: for all 2-d vectors

$\begin{vmatrix} \vec{u} \\ \vec{v} \end{vmatrix}$ is, up to sign, the area of

the parallelogram spanned by $\vec{u}, \vec{v}$



It is positive if you turn left to get from $\vec{u}$ to $\vec{v}$, and negative if you turn right to get from $\vec{u}$ to $\vec{v}$.

Geometrically, $\begin{vmatrix} \vec{u} \\ \vec{u} \end{vmatrix} = 0$ because the area is 0.

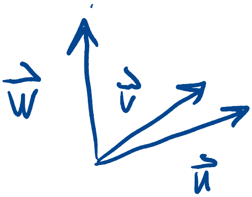
There is a 3-d version as well:

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$



$\begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$ is, up to sign,

the area of the parallelopiped spanned by $\vec{u}, \vec{v}, \vec{w}$. It is positive if right handed (Demonstrate)

Cross Product:

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

Whew! Why?

Now we multiply two vectors and obtain a vector in return. This is a very special 3-d operation.

$$\vec{a} \times \vec{a} = \vec{0}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

$$\begin{aligned} \vec{a} \cdot (\vec{a} \times \vec{b}) = {} & a_1(\cancel{a_2 b_3} - \cancel{a_3 b_2}) \\ & + a_2(\cancel{a_3 b_1} - \cancel{a_1 b_3}) \\ & + a_3(\cancel{a_1 b_2} - \cancel{a_2 b_1}) = 0 \end{aligned}$$

$\vec{a}$ is perpendicular to $\vec{a} \times \vec{b}$.

↑ geometry!

$$\vec{b} \cdot (\vec{a} \times \vec{b}) = -\vec{b} \cdot (\vec{b} \times \vec{a}) = -0 = 0.$$

$\vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ also.

Key Property: $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Mnemonic for computing, using determinants

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (a_2 b_3 - a_3 b_2)\,\vec{i}$$