12.3 The Dot Product

Def: $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Multiply two vectors, get a scalar.

This depends on having picked coordinates.

But: $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$.

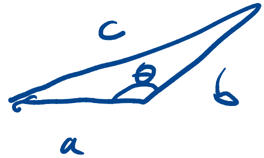
This depends only on the length scale.

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + |\vec{b}|^2$$

$$\vec{a} \cdot \vec{b} = \frac{1}{2} \left[|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a} - \vec{b}|^2 \right]$$

it depends <u>only</u> on the length scale.

Law of cosines:



$$2ab\cos\theta = a^2 + b^2 - c^2$$

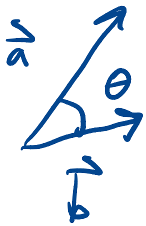
$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$0 \le \theta \le \pi$

$\uparrow$

$\theta = 0$, done

$\theta = \pi$, easy



$$\vec{i}\cdot\vec{j} = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$$

$$\vec{j}\cdot\vec{k} = 0\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0$$

$$\vec{k}\cdot\vec{i} = 0\cdot 1 + 0\cdot 0 + 1\cdot 0 = 0$$

$\cos\theta = 0$ in each case

$\Rightarrow \theta = \pi/2 \Rightarrow \perp$, perp

Fundamental Property of Dot Product:

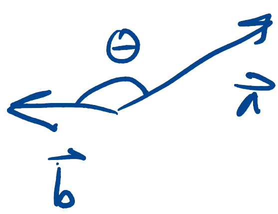
$$\vec{a} \cdot \vec{b} = 0 \quad \text{iff} \quad \vec{a} \perp \vec{b}$$

Secondary Property of Dot Product

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

$\vec{a} \cdot \vec{b} > 0$: $\cos\theta > 0$, acute

$\vec{a} \cdot \vec{b} < 0$: $\cos\theta < 0$, obtuse



E.g. Find the angle in radians between

$$\vec{a} = \langle 1, 2, 3 \rangle$$

$$\vec{b} = \langle -1, 2, 1 \rangle$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\vec{a} \cdot \vec{b} = -1 + 4 + 3 = 6$$

$$|\vec{a}|^2 = 1 + 4 + 9 = 14$$

$$|\vec{b}|^2 = 1 + 4 + 1 = 6$$

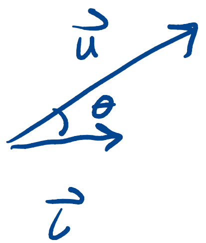
$$\cos\theta = \frac{6}{\sqrt{6}\,\sqrt{14}} = \sqrt{6/14}$$

$$\theta = \arccos\left(\sqrt{6/14}\right)$$

$$= 0.857 \text{ rad}$$

$$= 49.1^\circ$$

Direction cosines:



$$\vec{u} \cdot \vec{i} = |\vec{u}|\,|\vec{i}| \cos\theta$$

angle between vector and positive x-axis

we call $\cos\theta$ the direction cosine associated with the x-axis

(θ is direction angle, less common)

$$\vec{u} \cdot \vec{i} = |\vec{u}| \cos\theta$$

$$\vec{u} = \langle u_1, u_2, u_3 \rangle \Rightarrow \vec{u} \cdot \vec{i} = u_1$$

$$\vec{u} = \langle |\vec{u}| \cos\theta_x, |\vec{u}| \cos\theta_y, |\vec{u}| \cos\theta_z \rangle$$

$$= |\vec{u}| \langle \cos\theta_x, \cos\theta_y, \cos\theta_z \rangle$$

$$\frac{1}{|\vec{u}|}\vec{u} = \underbrace{\langle \cos\theta_x, \cos\theta_y, \cos\theta_z \rangle}$$

↳ unit vector of direction cosines

Text $\theta_x = \alpha$, $\theta_y = \beta$, $\theta_z = \gamma$.

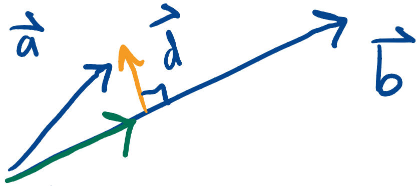
$$\vec{u} = \underbrace{|\vec{u}|\cos\theta_x}\,\vec{i} + |\vec{u}|\cos\theta_y\,\vec{j} + |\vec{u}|\cos\theta_z\,\vec{k}$$

↳ piece of $\vec{u}$ pointing along x-axis.

projection of $\vec{u}$ along x-axis.

Scalar projection of $\vec{u}$ along x-axis.

(Orthogonal) Projection



projection of $\vec{a}$ along $\vec{b}$

(write $\vec{a}$ as green + $\perp$ to $\vec{b}$)

$$\vec{a} = c\vec{b} + \vec{d} \qquad \vec{d} \perp \vec{b}$$

$$\vec{a} \cdot \vec{b} = c\,|\vec{b}|^2$$

$$c = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}$$

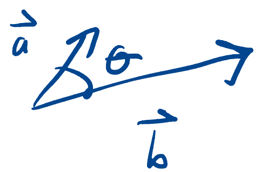
$$\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\,\vec{b} + \vec{d}$$

Def: $$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}$$

$$\underbrace{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}}\ \underbrace{\frac{\vec{b}}{|\vec{b}|}}_{\text{unit vector}}$$

$$\text{scal}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$\text{scal}_{\vec{b}}\,\vec{a} = |\vec{a}|\cos\theta$$



Dot Products + Physics

If a constant force $\vec{F}$ is applied to a body that moves from P to Q then the body gains/loses energy.

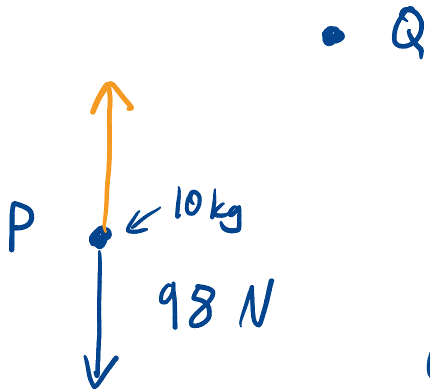
This change is the <u>work</u> done on the body.

work is a change in energy, a scalar

$$\text{work} = \underbrace{\vec{F}}_{kg\frac{m}{s^2} = N} \cdot \underbrace{\overrightarrow{PQ}}_{m}$$

$$[\text{work}] = kg\, m^2/s^2 = J, \quad \text{Joule}$$

unit of energy



P (1m, 3m)

Q (4m, 7m)

$$\vec{PQ} = \langle 3, 4 \rangle$$

$$\vec{F} = \langle 0, +98 \rangle$$

$$\vec{F} \cdot \vec{PQ} = +392 \text{ J}$$

I apply ≈ 400 J of energy to move from $\vec{P}$ to $\vec{Q}$.

See sled example