Newton's Method (II)

Math 426

University of Alaska Fairbanks

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Two Cases of Taylor's Theorem

Zeroth order (MVT):

$$f(x) = \overset{\text{approx}}{f(a)} + \overbrace{f'(\xi)(x-a)}^{\text{error term}}$$

ξ is between x and a.

First order: (Linear Approximation)

$$f(x) = \underbrace{\overbrace{f(a)}^{\text{number}} + \overbrace{f'(a)}^{\text{number}}(x-a)}_{\text{approx}} + \underbrace{\frac{1}{2} f''(\xi)(x-a)^2}_{\substack{\text{error term} \\ (\xi \text{ is between } x \text{ and } a)}}$$

Example

Suppose we approximate $\sin(x)$ by its first order Taylor polynomial centered at 0.

$a = 0$

$\sin(0)$

$$f(x) = \sin(x); \qquad f(0) = 0$$
$$f'(x) = \cos(x); \qquad f'(0) = 1$$
$$f''(x) = -\sin(x)$$

$\cos(0)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(\xi)(x-a)^2$$

$$\sin(x) = \sin(0) + \sin'(0)(x-0) + \frac{1}{2}\sin''(\xi)(x-0)^2$$

$$= 0 + 1 \cdot (x-0) + \frac{1}{2}\sin''(\xi)(x-0)^2$$

$$= x + \frac{1}{2}\sin''(\xi)x^2$$

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Taylor's theorem:

$$f(x) = f(0) + f'(0)(x-0) + \frac{1}{2} f''(\xi)(x-0)^2$$

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$$\sin(x) = x - \frac{1}{2}\sin(\xi)x$$

where ξ is some number between 0 and x

How big is the error?

$$\sin(x) = x - \frac{1}{2}\sin(\xi)x$$

First order Taylor polynomial:

$$P(x) = x$$

Remainder term:

$$R(x, \xi) = -\frac{1}{2}\sin(\xi)x^2$$

Suppose $|x| < 1/2$. How big is the error if we approximate $\sin(x)$ with its first order Taylor polynomial?

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Suppose $|x| < 1/2$. How big is the error if we approximate $\sin(x)$ with its first order Taylor polynomial?

$$\sin(x) = x - \frac{1}{2}\sin(\xi)\, x^2 \qquad |\sin(\xi)| \leq 1$$

$$|\sin(x) - x| = \frac{1}{2}|\sin(\xi)|\, x^2$$

$$\leq \frac{1}{2}x^2$$

If $|x| < \frac{1}{2}$ the error is $\leq \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

Taylor's Theorem

Suppose f has k continuous derivatives on $[a, \overset{x}{b}]$ and is $k+1$ times differentiable on $(a, \overset{x}{b})$. Then there exists $\xi \in (a, \overset{x}{b})$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(a)(b-a)^2 + \cdots + \frac{1}{k!} f^{(k)}(a)(b-a)^k +$$
$$+ \frac{1}{(k+1)!} f^{(k+1)}(\xi)(b-a)^{k+1}.$$

k^{th} order polynomial in b

Taylor polynomial:

approx: $f(\overset{x}{b}) \approx f(a) + f'(a)(\overset{x}{b}-a) + \cdots + \frac{1}{k!} f^{(k)}(a)(\overset{x}{b}-a)^k$

Remainder term:

$$\frac{1}{(k+1)!} f^{(k+1)}(\overset{\xi}{\cancel{a}})(\overset{x}{b}-a)$$

Example

Compute the third order Taylor polynomial of $f(x) = e^x$ centered at $x = 0$ and estimate the error in approximating $f(x)$ with it for $x \in [-1, 1]$. In particular, estimate the value of e and give an bound on the error.

$e = e^1 = f(1)$

$f(0) = e^0 = 1$

$f'(x) = e^x$; $f'(0) = e^0 = 1$

$f''(x) = e^x$; $f''(0) = 1$

$f'''(x) = e^x$; $f'''(0) = 1$

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$$f(0) = f'(0) = \cdots = f'''(0) = 1$$

$$f(x) = f(0) + f'(0)(x-0) + \frac{1}{2} f''(0)(x-0)^2 + \frac{1}{3!} f'''(0)(x-0)^3$$

$$+ f^{(4)}(\xi)\frac{1}{4!}(x-0)^4$$

$$f(x) = 1 + 1\cdot(x-0) + \frac{1}{2} 1\cdot(x-0)^2 + \frac{1}{3!} 1\cdot(x-0)^3 + \frac{1}{4!} x^4$$

$$+$$

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Taylor poly

$$P(x) = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3}_{3^{rd}\text{ order Taylor poly}}$$

error : $\frac{1}{4!}e^{\xi}x^4$

$$e^x - P(x) = \frac{1}{4!}e^{\xi}x^4$$

ξ is between 0 and x

Example

$$4! = 1\cdot 2\cdot 3\cdot 4$$
$$6\cdot 4 = 24$$

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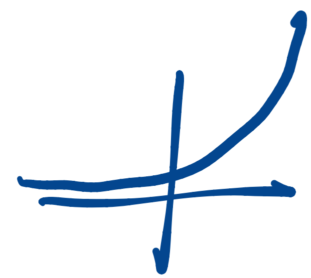

$$x \in [-1,1] \Rightarrow \xi \in [-1,1] \qquad |e^{\xi}| \le e^1 \le 3$$

$$|e^x - P(x)| \le \frac{1}{4!}\cdot 3x^4 = \frac{1}{\cancel{120}\ 24}\cdot 3x^4 = \frac{1}{\cancel{40}\ 8}x^4$$

Example

$$e = 2.667 \pm \frac{1}{8}$$

Compute the third order Taylor polynomial of $f(x) = e^x$ centered at $x = 0$ and estimate the error in approximating $f(x)$ with it for $x \in [-1, 1]$. In particular, estimate the value of e and give an bound on the error.

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$e^x = P(x) + \text{error}$$

if $x \in [-1, 1]$ $\quad |\text{error}| \le \frac{1}{8} x^4$

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \varepsilon = 2.6667 + \varepsilon$$

$$|\varepsilon| \le \frac{1}{8} \qquad 1^4 = 1$$

Linear Approximation

Taylor (first order):

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(\xi)(x-a)^2.$$

Linear approximation:

$L(x)$ is linear and

$L(a) = f(a)$

$L'(a) = f'(a)$

$$P(x) = f(a) + f'(a)(x-a)$$

$L(x)$

"linear"

$$L(a) = f(a) + f'(a)(a-a)$$
$$= f(a) + f'(a) \cdot 0$$
$$= f(a)$$

$$\frac{d}{dx} L(x) = 0 + f'(a) \cdot 1$$
$$= f'(a)$$
$$L'(a) = f'(a)$$

Linear Approximation

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Linear approximation:

$$P(x) = f(a) + f'(a)(x-a)$$

Key properties:

$$P(a) = f(a) + f'(a)(a-a) = f(a)$$

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Derivative (with respect to x!)

$$P'(x) = 0 + f'(a)(1-0) = f'(a)$$

So $P'(a) = f'(a)$.

Linear Approximation

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Linear approximation:

$$P(x) = f(a) + f'(a)(x-a)$$

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Derivative (with respect to x!)

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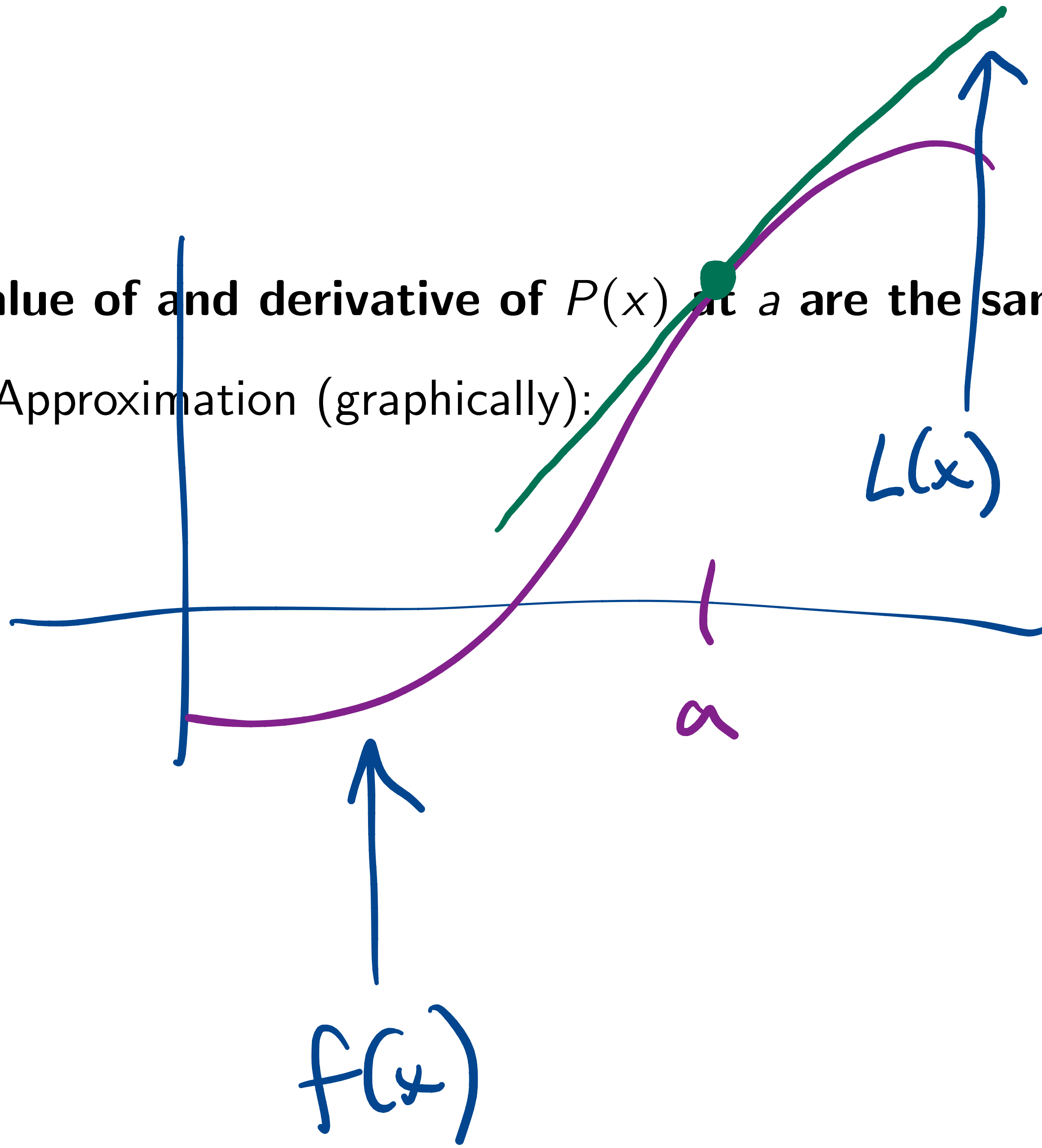
The value of and derivative of $\cancel{P}$ L(x) at a are the same as f's

Linear Approximation

The value of and derivative of $P(x)$ at a are the same as f's

Linear Approximation (graphically):

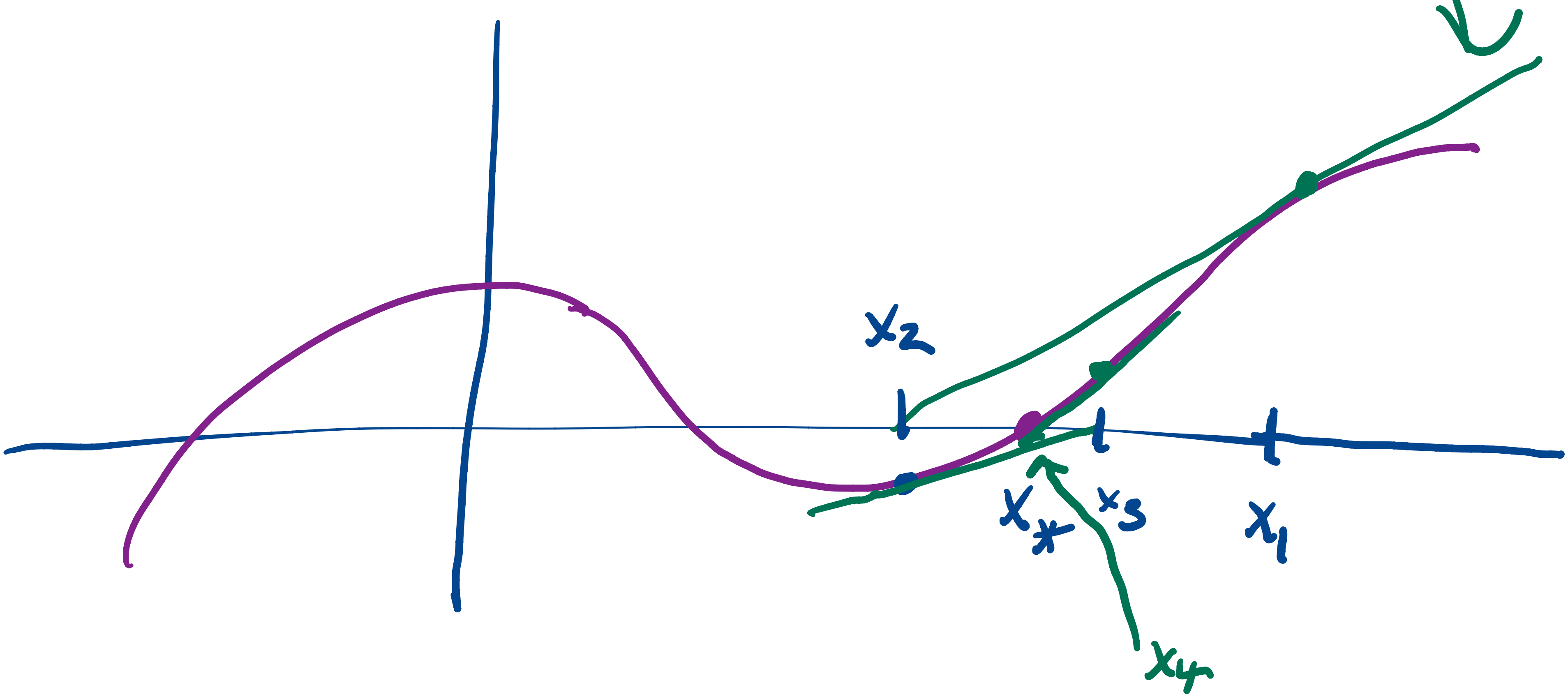


Newton's Method

$f(x) = 0$?

$L(x) = 0$

Want to solve $f(x) = 0$. Use a linear approximation instead!



Newton's Method (Formula)

Want to solve $f(x) = 0$.

Initial guess x_1

Linear approximation

$$L(x) = f(x_1) + f'(x_1)(x - x_1)$$

$L(x) = 0$? $\quad f(x_1) + f'(x_1)(x - x_1) = 0$

$$f'(x_1)(x - x_1) = -f(x_1)$$

new guess

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton's Method (Formula)

new guess

$$x = \boxed{x_1 - \frac{f(x_1)}{f'(x_1)}}$$

Now repeat. Use linearization at x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\vdots$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$