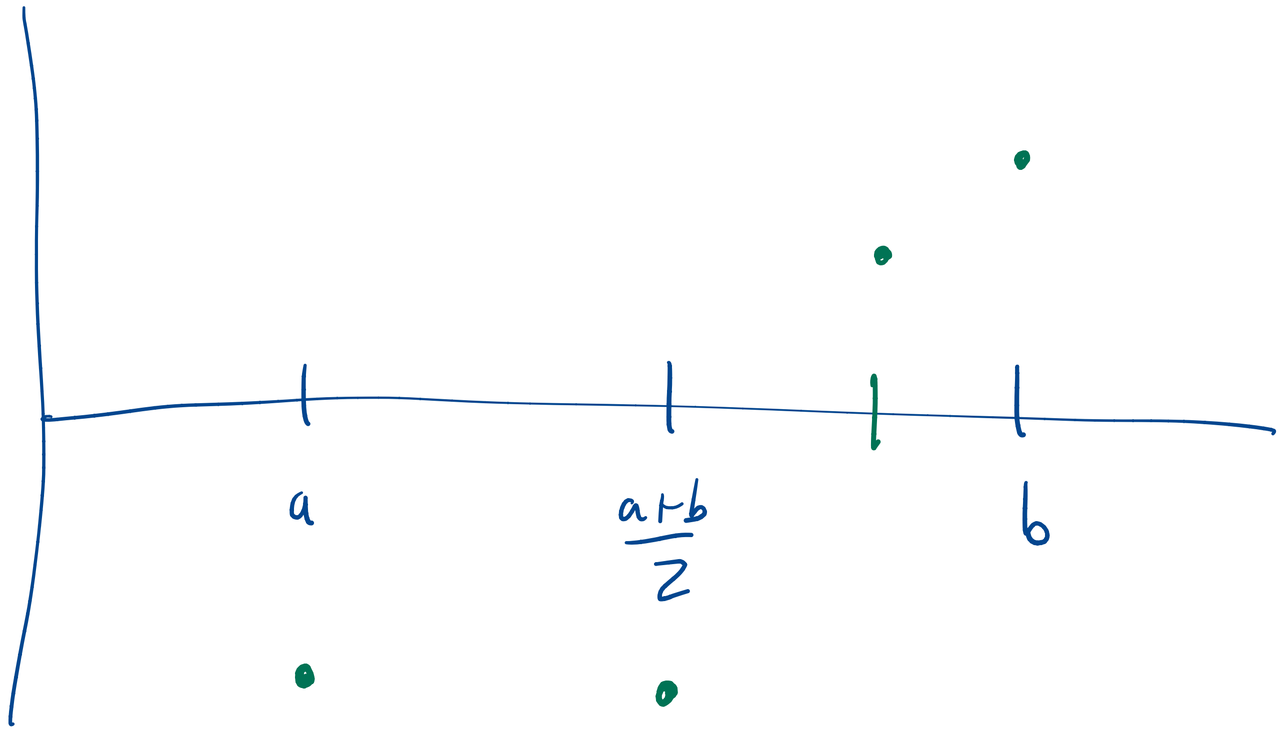
Recall

$f(x) \longrightarrow$ continuous

$f(x) = 0$



Analysis of Bisection

Given an application of bisection:

1. How good an approximation is the result?
2. How much work is needed to compute the result?

Analysis of Bisection

$f(x_{exact}) = 0$

Given an application of bisection:

1. How good an approximation is the result?
2. How much work is needed to compute the result?

Notation:

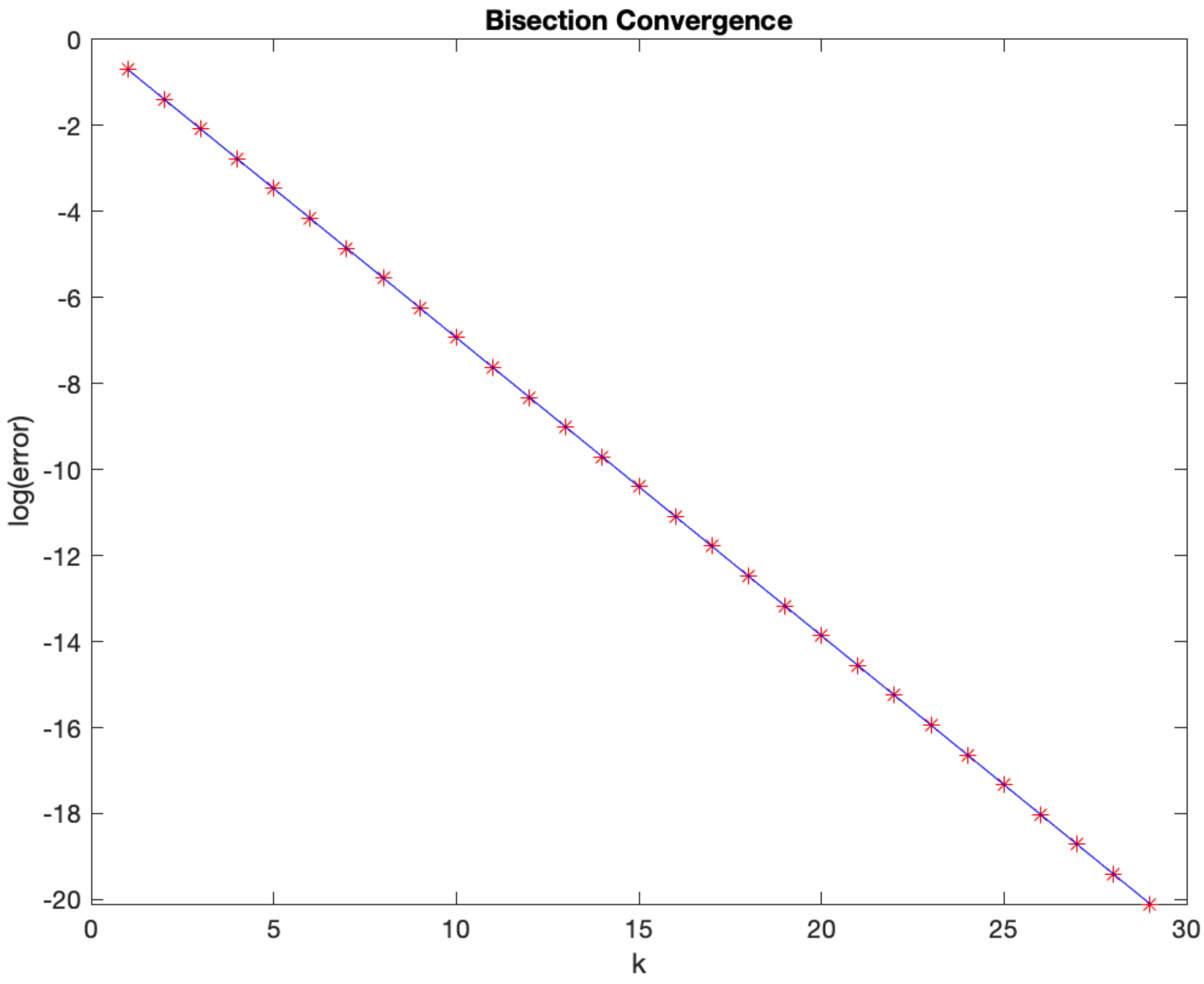
- a_k, b_k: the interval endpoints at step k
- $m_k = (a_k + b_k)/2$: the midpoint of interval k
- $e_k = |x_{\text{exact}} - m_k|$: the **absolute error** at step k

approximation of root

$e_k = x_{exact} - m_k$

$|e_k|$

Convergence Rate



Convergence Rate

$$\log(|err|) = mk + b$$

$\ln$ $\quad \log_2 \to \log 2$

$\log_{10} \to \log 10$

$$|err| = e^{mk+b}$$

$$= e^{mk} \cdot e^{b}$$

$$= (e^b)(e^m)^k$$

$$= C\alpha^k$$

$$|err| \sim C\alpha^k$$

$$m = \log(1/2)\ ?$$

$$\alpha = e^m$$

$$= e^{\log(1/2)}$$

$$= \frac{1}{2}\ (!)$$

$$|err| \sim C\left(\frac{1}{2}\right)^k$$

Convergence Rate

$$|b_2 - a_2| = \frac{1}{2} |b_1 - a_1|$$

$$|b_3 - a_3| = \frac{1}{2} |b_2 - a_2| = \frac{1}{2} \frac{1}{2} |b_1 - a_1|$$

$$= \left(\frac{1}{2}\right)^2 |b_1 - a_1|$$

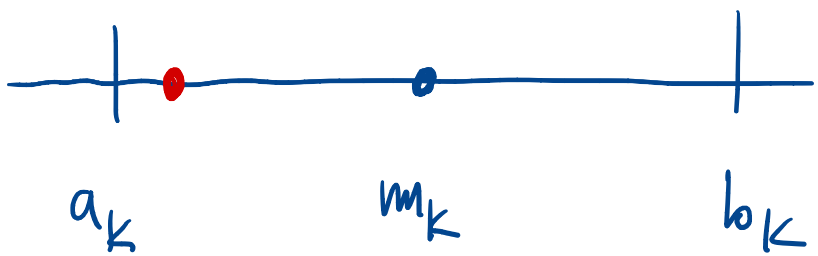
$$|b_4 - a_4| = \left(\frac{1}{2}\right)^3 |b_1 - a_1|$$

$$|b_k - a_k| = \left(\frac{1}{2}\right)^{k-1} |b_1 - a_1|$$

Convergence Rate

$$|b_k - a_k| = \left(\frac{1}{2}\right)^{k-1} |b_1 - a_1|$$

error at step k vs. $|b_k - a_k|$?



$$\text{error} \leq \frac{1}{2} |b_k - a_k|$$

$$\text{error} \leq \frac{1}{2} \left(\frac{1}{2}\right)^{k-1} |b_1 - a_1|$$

$$= \left(\frac{1}{2}\right)^{k} |b_1 - a_1|$$

Labor per Digit

Gaining a digit of accuracy means

decreasing the error by a factor of $\frac{1}{10}$

$$\left(\frac{1}{2}\right)^{\beta} = \frac{1}{10}$$

$$\log_2\left(\left(\frac{1}{2}\right)^{\beta}\right) = \log_2\left(1/10\right)$$

$$\beta \cdot (-1) = \log_2\left(1/10\right)$$

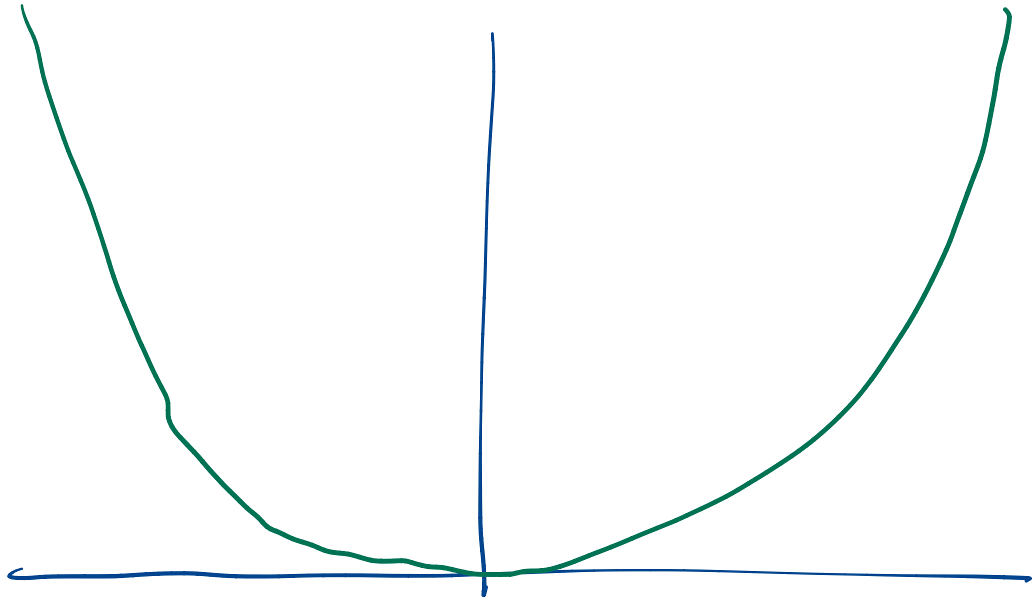
$$\beta = \underbrace{\log_2(10)}_{\approx 3.3}$$

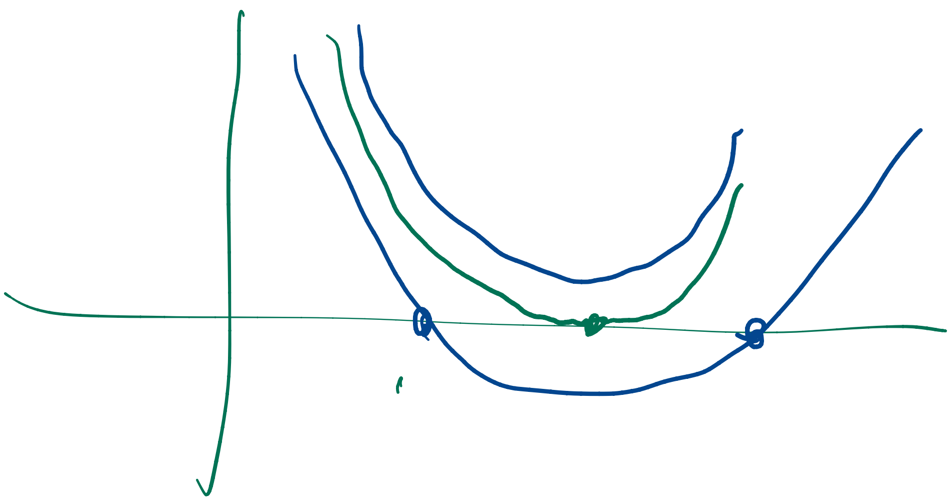
$$\left(\frac{1}{2}\right)^{3.3} = \frac{1}{10}$$

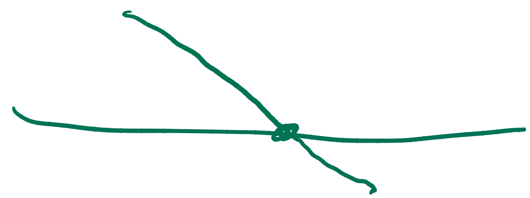
~~Labor per Digit~~

$f(x) = x^2$

$f(0) = 0$







What can go wrong?

- Need to have a guess for the initial interval.
- Some (rare) root cannot be found: $F(x) = x^2$ never changes sign.
- Workload seems fair: every 3.3 steps we gain a digit. But we can do better!

Newton's Method (I)

Math 426

University of Alaska Fairbanks

September 4, 2020

Mean Value Theorem

Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

slope is

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

MVT Rewritten

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

$$f(b) = f(a) + f'(\xi) \cdot (b - a)$$

$$f(x) = \underbrace{f(a)}_{\text{approximation}} + \underbrace{f'(\xi) \cdot (x - a)}_{\text{error}}$$