LU Factorization

Math 426

University of Alaska Fairbanks

September 30, 2020

Column perspecive on multiplication

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a_2} & \cdots & \mathbf{a}_n \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A\mathbf{x} = x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n}$$

Column perspective on multiplication

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$$A\mathbf{x} = x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n}$$

$$B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{pmatrix}$$

Then:

$$AB = \begin{pmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_k \end{pmatrix}$$

Row perspective on multiplication

$A\vec{x}$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}^T = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}$$

$$B = \begin{pmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{pmatrix}$$

$$\mathbf{y}^T = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}$$

$$\mathbf{y}^T B = y_1 \mathbf{r}_1^T + \cdots + y_n \mathbf{r}_n^T$$

$$B = \begin{pmatrix} r_{11} & \cdots & r_{1k} \\ r_{21} & & r_{2k} \\ \vdots & & \\ r_{m1} & & r_{mk} \end{pmatrix}$$

Row perspective on multiplication

$$B = \begin{pmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{pmatrix}$$

$$\mathbf{y}^T = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix}$$

$$AB \downarrow \begin{pmatrix} y_1^T \\ \vdots \\ y_m^T \end{pmatrix}$$

$$\mathbf{y}^T B = y_1 \mathbf{r}_1^T + \cdots + y_n \mathbf{r}_n^T$$

$$A = \begin{pmatrix} \mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_m^T \end{pmatrix}$$

$$AB = \begin{pmatrix} y_1^T B \\ \vdots \\ y_m^T B \end{pmatrix}$$

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Sample Linear Algebra Problem

Find x_1, x_2 and x_3 such that

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 0x_3 &= 2 \end{aligned}$$

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$$
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\end{aligned}
$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Want to solve $A\mathbf{x} = \mathbf{b}$.

Sample Linear Algebra Problem

Find x_1, x_2 and x_3 such that

$A^{-1} = (\vec{b}_1, \ldots, \vec{b}_n)$

$$x_1 + 2x_2 + 3x_3 = 1$$
$$4x_1 + 5x_2 + 6x_3 = 0$$
$$7x_1 + 8x_2 + 0x_3 = 2$$

$A\vec{x} = \vec{b}$

$AA^{-1} = I$

$\underbrace{A^{-1}A}\vec{x} = A^{-1}\vec{b}$

$(A\vec{b}_1, \ldots, A\vec{b}_n) = I$

A find A^{-1}

$\vec{x} = A^{-1}\vec{b}$

$A\vec{b}_k = \vec{e}_k$

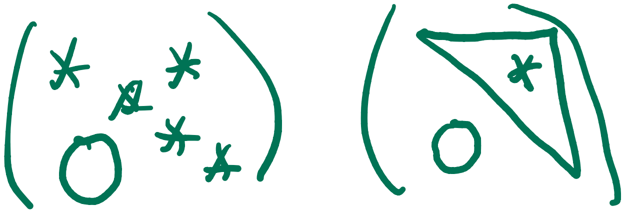
column k of I.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Want to solve $A\mathbf{x} = \mathbf{b}$.

If A has an inverse, $A^{-1}A\mathbf{x} = A^{-1}\mathbf{x}$ so $x = A^{-1}\mathbf{x}$. But finding A^{-1} is about n-times harder than solving $A\mathbf{x} = \mathbf{b}$!

Gaussian Elimination (Step 1: row reduction)



upper triangular

$L_1 A$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}$$

Perform row operations on A and b simultaneously to reduce A to upper triangular form:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -21 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{pmatrix} = U$$

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -4 \\ -5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \vec{b}'$$

Gaussian Elimination (Step 2: back substitution)

$$U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{pmatrix}; \qquad \mathbf{b}' = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$$

Solve $U\mathbf{x} = \mathbf{b}'$ instead.

$x_1 + 2x_2 + 3x_3 = 1 \Rightarrow x_1 = -2$

$-3x_2 - 6x_3 = -4 \Rightarrow -3x_2 = -4 + 6x_3 = -6$

$\Rightarrow x_2 = 2$

$-9x_3 = 3 \Rightarrow x_3 = -1/3$

Row operations via matrix multiplication

Recall: in the first step of row reduction we added -4 copies of row 1 of A to row 2 and -7 copies of row 1 to row 3.

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$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix} \qquad L_1 A = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} A \\ \begin{pmatrix} -4 & 1 & 0 \end{pmatrix} A \\ \begin{pmatrix} -7 & 0 & 1 \end{pmatrix} A \end{pmatrix}$$

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$$\begin{pmatrix} -4 & 1 & 0 \end{pmatrix} A = ?$$

$-4 \cdot \text{row}_1 + 1 \cdot \text{row}_2 + 0 \cdot \text{row}_3$

Row operations via matrix multiplication

$$L_1 = \begin{pmatrix} 1 & & & 0 \\ * & 1 & & \\ * & 0 & 1 & \\ * & 0 & 0 & 1 \end{pmatrix}; \quad L_2 = \begin{pmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & * & 1 & \\ 0 & * & 0 & 1 \end{pmatrix}$$

Recall: in the second step of row reduction we added -2 copies of row 2 of $L_1 A$ to row 3.

$$L_3 = \begin{pmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & * & 1 \end{pmatrix}$$

$$U = L_2 L_1 A$$

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} (L_1 A)$$

Row operations via matrix multiplication

Recall: in the second step of row reduction we added -2 copies of row 2 of $L_1 A$ to row 3.

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad U = L_2 L_1 A = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} L_1 A \\ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} L_1 A \\ \begin{pmatrix} 0 & -2 & 1 \end{pmatrix} L_1 A \end{pmatrix}$$

Tidy Gaussian Elimination

$$A\mathbf{x} = \mathbf{b}$$
$$L_1 A\mathbf{x} = L_1\mathbf{b}$$
$$L_2 L_1 A\mathbf{x} = L_2 L_1\mathbf{b}$$

$U = L_2 L_1 A$ is upper triangular
$b' = L_2 L_1 \mathbf{b}$.

$Ux = b'$

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New system to solve:

$$U\mathbf{x} = \mathbf{b}'$$

Inverses of L matrices

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix} \qquad L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Just change signs!

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Just change signs!

$$L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Why Gaussian Elimination Works

If $Ax = b$

then

$Ux = b'$

$$L_2 L_1 A = U; \qquad L_2 L_1 \mathbf{b} = \mathbf{b}'$$

Why Gaussian Elimination Works

$$L_2 L_1 A = U; \qquad L_2 L_1 \mathbf{b} = \mathbf{b}'$$

Suppose $\mathbf{x}$ solves $U\mathbf{x} = \mathbf{b}'$.

Why Gaussian Elimination Works

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Suppose $\mathbf{x}$ solves $U\mathbf{x} = \mathbf{b}'$.

$$L_1 A = L_2^{-1} U$$

$$\begin{aligned}
U\mathbf{x} &= \mathbf{b}' \\
L_2^{-1} U\mathbf{x} &= L_2^{-1}\mathbf{b}' \\
L_1^{-1} L_2^{-1} U\mathbf{x} &= L_1^{-1} L_2^{-1}\mathbf{b}' \\
L_1^{-1} L_2^{-1} L_2 L_1 A\mathbf{x} &= L_1^{-1} L_2^{-1} L_2 L_1 \mathbf{b} \\
A\mathbf{x} &= \mathbf{b}
\end{aligned}$$

$$A = \underbrace{L_1^{-1} L_2^{-1}}_{L} U$$

$$A = L U$$

Why Gaussian Elimination Works

$$L_2 L_1 A = U; \qquad L_2 L_1 \mathbf{b} = \mathbf{b}'$$

Suppose $\mathbf{x}$ solves $U\mathbf{x} = \mathbf{b}'$.

$$\begin{aligned}
U\mathbf{x} &= \mathbf{b}' \\
L_2^{-1} U\mathbf{x} &= L_2^{-1}\mathbf{b}' \\
L_1^{-1} L_2^{-1} U\mathbf{x} &= L_1^{-1} L_2^{-1}\mathbf{b}' \\
L_1^{-1} L_2^{-1} L_2 L_1 A\mathbf{x} &= L_1^{-1} L_2^{-1} L_2 L_1 \mathbf{b} \\
A\mathbf{x} &= \mathbf{b}
\end{aligned}$$

Let $L = L_1^{-1} L_2^{-1}$. Then

$$A = LU$$

is the LU factorization of A.

LU Factorization

$$A = LU$$

where U is upper triangular, L is lower triangular and L has 1's on the diagonal.

$$L = L_1^{-1} L_2^{-1}$$

$$L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 0 & 1 \end{pmatrix}$$

LU Factorization

$$A = LU$$

where U is upper triangular, L is lower triangular and L has 1's on the diagonal.

$$L = L_1^{-1} L_2^{-1}$$

A mathematical miracle:

$$L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

Up to sign, L just contains the factors used in Gaussian elimination.

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LU Factorization

$$A = LU$$

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Up to sign, L just contains the factors used in Gaussian elimination. And U is the final reduction matrix from Gaussian elimination. **If you can do Gaussian elimination, then you can do LU factorization.**

Discussion of mathematical miracle

$L_2 L_1 A$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{pmatrix} \qquad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$L = L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{pmatrix}$$

vs:

$$L_2 L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ \mathbf{1} & -2 & 1 \end{pmatrix}$$

Solution via LU factorization

Want to solve

$$A\mathbf{x} = \mathbf{b}.$$

Solution via LU factorization

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$$A\mathbf{x} = \mathbf{b}.$$

1. Perform an LU factorization of A. Now want to solve

$$LU\mathbf{x} = \mathbf{b}$$

Solution via LU factorization

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1. Perform an LU factorization of A. Now want to solve

$$LU\mathbf{x} = \mathbf{b}$$

$Ux = b'$

2. Solve

$$L\mathbf{b}' = \mathbf{b}$$

   via back substitution.

Solution via LU factorization

Want to solve

$[L]$

$$A\mathbf{x} = \mathbf{b}.$$

$$[\ast \mid b] \to [U \mid b']$$

$$LUx = b$$

1. Perform an LU factorization of A. Now want to solve
$$LU\mathbf{x} = \mathbf{b}$$
2. Solve
$$L\mathbf{b}' = \mathbf{b}$$
via back substitution.
3. Now want to solve
$$LU\mathbf{x} = L\mathbf{b}' \Rightarrow Ux = b'$$
or equivalently
$$U\mathbf{x} = \mathbf{b}'.$$

$$LU \to A\backslash b \Rightarrow x$$