Linear Algebra

Math 426

University of Alaska Fairbanks

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28

$a \oplus b$

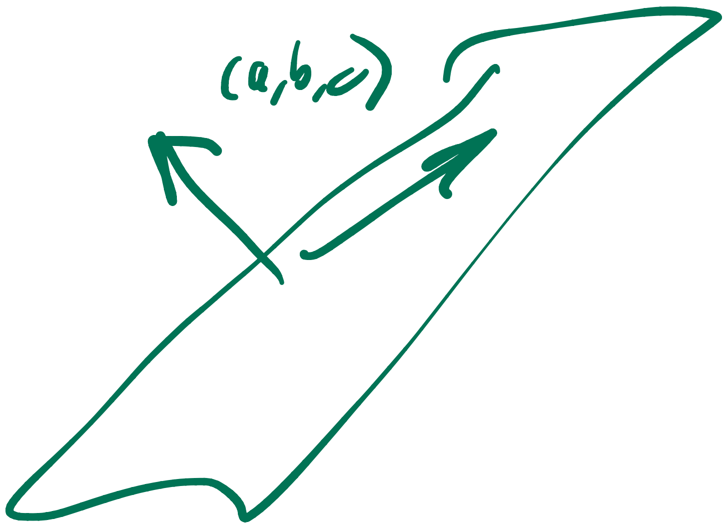
Sample problem

Given numbers: a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2.

Unknowns: x, y, z

$$
\begin{aligned}
a_1 x + b_1 y + c_1 z &= d_1 \\
a_2 x + b_2 y + c_2 z &= d_2
\end{aligned}
$$

Geometric perspective



$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$

The equation $ax + by + cz = d$ is satisfied by points (x, y, z) lying in a plane. The normal of the plane is (a, b, c). When $d = 0$ the plane passes through the origin. Otherwise, d determines a different parallel plane.

A point (x, y, z) solves the system if it is on the intersection of the given planes.

Linear combination perspective

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$x\vec{a} + y\vec{b} + z\vec{c} = d$$

$$\begin{bmatrix} a_1 x \\ a_2 x \end{bmatrix} + \begin{bmatrix} b_1 y \\ b_2 y \end{bmatrix} + \begin{bmatrix} c_1 z \\ c_2 z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Linear combination perspective

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Find x, y, z satisfying

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$$

Linear combination perspective

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Find x, y, z satisfying

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$$

An expression of the type on the left-hand side is called a **linear combination** of the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Find a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ that equals $\mathbf{d}$.

Matrix-vector multiplication: shorthand for linear combinations

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}; \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}; \qquad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}; \qquad \mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

By definition

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

2x3 matrix

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

$A \Rightarrow 5 \times 7$ matrix

$\vec{y} \Rightarrow$ vector

Notation:

$$A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix}; \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$A\vec{y}$

Find a solution $\mathbf{x}$ of

$$A\mathbf{x} = \mathbf{d}.$$

Find a linear combination of the columns of A that equals $\mathbf{d}$.

Column perspective of matrix-vector multiplication

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a_2} & \cdots & \mathbf{a}_n \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$A\mathbf{x} = x_1\mathbf{a_1} + \cdots + x_n\mathbf{a_n}$$

For this to make sense, the number of columns of A has to match the number of rows of $\mathbf{x}$.

Matrix-matrix multiplication (column perspective)

$m \times n$ $\qquad\qquad\qquad\qquad$ $n \times k$

$$A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix} \qquad B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{pmatrix}$$

Then:

$$AB = \begin{pmatrix} A\mathbf{b}_1 & \cdots & A\mathbf{b}_k \end{pmatrix}$$

This only works if the number of columns of A matches the number of rows of B. An $m \times n$ matrix multiplied by a $n \times k$ matrix yields a

$m \times k$ $\qquad\qquad$ $(m \times n)\,(n \times k)$

$m \times k$

Matrix-matrix multiplication (column perspective)

$$A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix} \qquad B = \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{pmatrix}$$

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This only works if the number of columns of A matches the number of rows of B. An $m \times n$ matrix multiplied by a $n \times k$ matrix yields a $\boxed{m \times k \text{ matrix}}$.

Matrix-matrix multiplication (row perspective)

$1 \times 3 \quad 3 \times 3 \rightarrow 1 \times 3$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\left((x, y, z)\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} \quad (x\ y\ z)\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \quad (x\ y\ z)\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} \right)$$

$$\left(xa_1 + yb_1 + zc_1,\ xa_2 + yb_2 + zc_2,\ xa_3 + yb_3 + zc_3 \right)$$

Matrix-matrix multiplication (row perspective)

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$(xa_1 + yb_1 + zc_1,\ xa_2 + yb_2 + zc_2,\ xa_3 + yb_3 + zc_3)$$

$$x(a_1, a_2, a_3) + y(b_1, b_2, b_3) + z(c_1, c_2, c_3)$$

Matrix-matrix multiplication (row perspective)

$$A(\vec{b}_1, \ldots, \vec{b}_n) = (A\vec{b}_1, \ldots, A\vec{b}_p)$$

$$\underbrace{\begin{pmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{pmatrix}}_{A} B = \begin{pmatrix} \mathbf{r}_1^T B \\ \vdots \\ \mathbf{r}_m^T B \end{pmatrix}$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$r_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad r_1^T = \begin{pmatrix} a & b & c \end{pmatrix}$$

Identity Matrices

$n \times n$ identity I_n

eye(n)

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$I_n \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

Identity Matrices

$n \times n$ identity I_n

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$I_n \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

$$I_n\mathbf{x} = \mathbf{x}$$

$I_n B = B$

If $B = [\mathbf{b}_1, \cdots, \mathbf{b}_{\cancel{p}\,k}]$ $n \times k$

$$I_n B = [I_n\mathbf{b}_1, \cdots, I_n\mathbf{b}_{\cancel{p}\,k}] = [\mathbf{b}_1, \cdots, \mathbf{b}_{\cancel{p}\,k}]$$

Identity Matrices

$n \times n$ identity I_n

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$I_n \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

Use the row perspective of matrix multiplication to show

$$BI_n = B$$

whenever B is $(k \times n)$

Identity Matrices

$m \times j$ $\quad w \times w$

$n \times n$ identity I_n

$n \times k$

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$\underline{A}\,\underline{C} = I$$

$$CA = I$$

$$C = A^{-1}$$

$$I_n \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$$

Use the row perspective of matrix multiplication to show

$$BI_n = B$$

$$A^{-1} A x = I x$$

whenever B is $(k \times n)$

$$= x$$

If A and C are $n \times n$ we say C is the inverse matrix of A if $AC = CA = I_n$. We write $C = A^{-1}$.

Gaussian Elimination (Step 1)

$A^{-1} = (b_1, \ldots, b_3)$ $\quad AA^{-1} = (Ab_1, \ldots, Ab_3)$

$= (e_1, \ldots, e_3)$

$Ab_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$Ab_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$Ab_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 30 \end{pmatrix}; \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Want to solve $A\mathbf{x} = \mathbf{b}$.

Find A^{-1}!

$Ax = b$

$\underbrace{A^{-1}A}\, x = A^{-1}b$

$x = A^{-1}b$