Fixed Point Convergence

Math 426

University of Alaska Fairbanks

September 16, 2020

Fixed Point Iteration

$$\Phi(x) = x \qquad \Phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\Phi(x) = cx$$

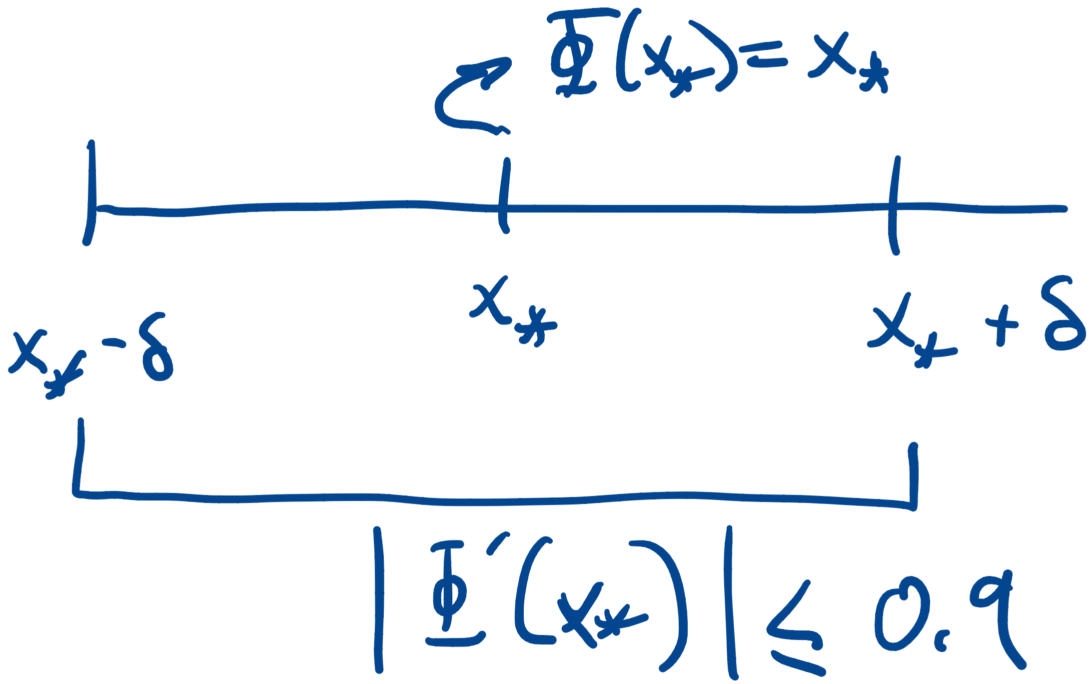
$$\Phi(\Phi(x)) = c(cx)$$

$$x_{k+1} = \Phi(x_k) \Rightarrow x_k = c^{k-1} x_1$$

$$x_k \to 0 \quad \text{when} \quad |c| < 1$$

Generalization

Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \leq 0.9$ on $[x_* - \delta, x_* + \delta]$.



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Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \le 0.9$ on $[x_* - \delta, x_* + \delta]$.

Baby Taylor (MVT)

ξ is between x and a

$$f(x) = f(a) + f'(\xi)(x - a)$$

Generalization

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Baby Taylor (MVT)

$$f(x) = f(a) + f'(\xi)(x - a)$$

$f \to \Phi$ $\quad$ $x \to x_1$

$a \to x_*$

$$\Phi(x_1) = \Phi(x_*) + \Phi'(\xi_1)(x_1 - x_*)$$

Generalization

Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \leq 0.9$ on $[x_* - \delta, x_* + \delta]$.

Baby Taylor (MVT)

$\Phi(x_1) = x_2$

$$f(x) = f(a) + f'(\xi)(x - a)$$

$$\Phi(x_1) = \Phi(x_*) + \Phi'(\xi_2)(x_1 - x_*)$$

$$\underbrace{\Phi(x_1)}_{x_2} - x_* = \underbrace{\Phi(x_*) - x_*}_{0} + \Phi'(\xi_2)\underbrace{(x_1 - x_*)}_{-e_1}$$

$x_2 - x_*$

$-e_2$

Generalization

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Baby Taylor (MVT)

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$$\underbrace{x_2 - x_*}_{-e_2} = \Phi'(\xi_2)\underbrace{(x_1 - x_*)}_{-e_1}$$

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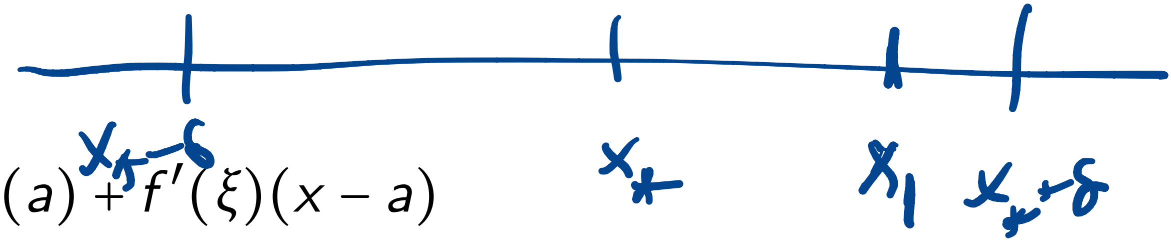

$$f(x) = f(a) + f'(\xi)(x - a)$$

$$\Phi(x_1) = \Phi(x_*) + \Phi'(\xi_2)(x_1 - x_*)$$

$$\Phi(x_1) - x_* = \Phi(x_*) - x_* + \Phi'(\xi_2)(x_1 - x_*)$$

$$x_2 - x_* = \Phi'(\xi_2)(x_1 - x_*)$$

$$|e_2| = \overbrace{|\Phi'(\xi_1)|}^{\le 0.9} |e_1|$$

$$e_2 = \Phi'(\xi_1) e_1$$

Convergence of Fixed Point Methods

Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \leq 0.9$ on $[x_* - \delta, x_* + \delta]$.

Then

$$e_2 = \Phi'(\xi_2)e_1$$

where ξ_2 is between x_* and x_1.

Convergence of Fixed Point Methods

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Then

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where ξ_2 is between x_* and x_1.

Observe: $x \in [x_* - \delta, x_* + \delta]$ if and only if $|x - x_*| \leq \delta$.

$$-\delta \leq x - x_* \leq \delta$$

$$x_* - \delta \leq x \leq x_* + \delta$$

Convergence of Fixed Point Methods

Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \leq 0.9$ on $[x_* - \delta, x_* + \delta]$.

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Observe: $x \in [x_* - \delta, x_* + \delta]$ if and only if $|x - x_*| \leq \delta$.

If $x_1 \in [x_* - \delta, x_* + \delta]$ then so is ξ_1. So

$$|x_2 - x_*| = |e_2| = |\Phi'(\xi_2)||e_1| \leq 0.9|x_1 - x_*| \leq 0.9\delta.$$

Convergence of Fixed Point Methods

Suppose $\Phi(x_*) = x_*$ and $|\Phi'(x)| \le 0.9$ on $[x_* - \delta, x_* + \delta]$.

Then

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$$|x_2 - x_*| = |e_2| = |\Phi'(\xi_2)||e_1| \le 0.9|x_1 - x_*| \le 0.9\delta. \quad \le \delta$$

Key conclusions! If $x_1 \in [x_* - \delta, x_* + \delta]$ then

1. $x_2 \in [x_* - \delta, x_* + \delta]$
2. $|e_2| \le 0.9|e_1|$

Convergence of Fixed Point Methods

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Now rinse and repeat. Since $x_2 \in [x_* - \delta, x_* + \delta]$

1. $x_3 \in [x_* - \delta, x_* + \delta]$
2. $|e_3| \leq 0.9|e_2| \leq (0.9)^2|e_1| \leq (0.9)^2\delta$

Convergence of Fixed Point Methods

Key conclusions! If $x_1 \in [x_* - \delta, x_* + \delta]$ then

1. $x_2 \in [x_* - \delta, x_* + \delta]$
2. $|e_2| \le 0.9|e_1|$

$|\Phi'(x)| \le 0.9$

on $[x_* - \delta, x_* + \delta]$

Now rinse and repeat. Since $x_2 \in [x_* - \delta, x_* + \delta]$

1. $x_3 \in [x_* - \delta, x_* + \delta]$
2. $|e_3| \le 0.9|e_2| \le (0.9)^2|e_1| \le (0.9)^2\delta$

0.5

If $x_1 \in [x_* - \delta, x_* + \delta]$ then

Replace 0.9 with

1. Each $x_k \in [x_* - \delta, x_* + \delta]$.
2. $|e_k| \le (0.9)^k\delta$, so $e_k \to 0$.

$x_k \to x_*$

c, $0 \le c < 1$

and all is ok.

Convergence of Fixed Point Methods

Key conclusions! If $x_1 \in [x_* - \delta, x_* + \delta]$ then

1. $x_2 \in [x_* - \delta, x_* + \delta]$
2. $|e_2| \leq 0.9|e_1|$

Now rinse and repeat. Since $x_2 \in [x_* - \delta, x_* + \delta]$

1. $x_3 \in [x_* - \delta, x_* + \delta]$
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If $x_1 \in [x_* - \delta, x_* + \delta]$ then

1. Each $x_k \in [x_* - \delta, x_* + \delta]$.
2. $|e_k| \leq (0.9)^k\delta$, so $e_k \to 0$.

What happens if we replace 0.9 by other numbers?

Fixed Point Convergence Theorem

Theorem

Suppose that Φ is continuously differentiable on an interval $[x_* - \delta, x_* + \delta]$ centered around a fixed point x_* and that $|\Phi'(x)| < 1$ for all x in the interval. Then fixed point iteration converges to x_* starting from any initial point in the interval.

$$|\Phi'(x)| \leq c < 1$$

$$\Phi(x) = cx$$

$$|\Phi'(x)| = |c|$$

Newton's Method Convergence

$f(x_*) = 0$

x_*

$\Phi(x_*) = x_*$

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\Phi'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$\Phi'(x_*) = 0$

Assuming $f'(x_*) \neq 0$,

$$\Phi'(x_*) = 0$$

and by continuity of Φ', it remains near 0 on a small interval around x_*.

$$\Phi'(x_*) = \left. \frac{f(x_*)f''(x_*)}{f'(x_*)^2} \right] = 0$$

Newton's Method Convergence

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

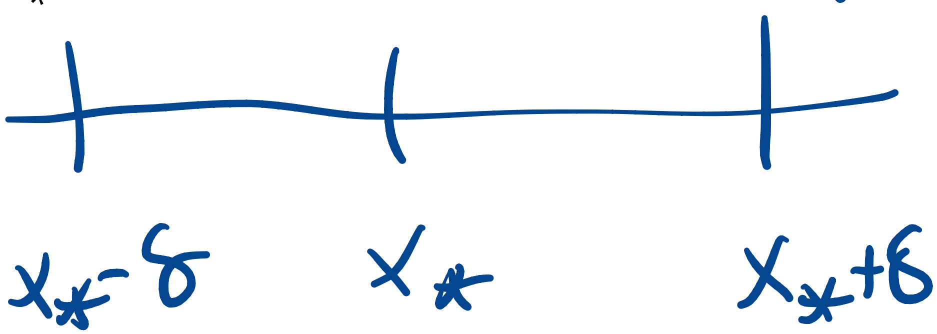
$$\Phi'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

Assuming $f'(x_*) \neq 0$,

$$\Phi'(x_*) = 0$$

$\Phi'(x_*) \approx 0$

and by continuity of Φ', it remains near 0 on a small interval around x_*.



$\Phi'(x_*) = 0$

Newton's Method Convergence

Theorem

Suppose $f \in C^2(\mathbb{R})$ *and* $f(x_*) = 0$. *If* $f'(x_*) \neq 0$ *then there is an* $\delta > 0$ *such that if* $x_1 \in (x_* - \delta, x_* + \delta)$ *then*

1. $x_k \to x_*$
2. $\dfrac{|e_{k+1}|}{|e_k|^2} \to \left| \dfrac{f''(x_*)}{2f'(x_*)} \right|$

Computer Rep of Numbers

~~Fixed Point Convergence~~

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Binary/Decimal Represesntation of Integers

386_{10}

$$3 \times 10^2 + 8 \times 10^1 + 6 \times 10^0$$

$$300 \qquad 80 \qquad 6$$

Binary/Decimal Represesntation of Integers

386_{10}

$$3 \times 10^2 + 8 \times 10^1 + 6 \times 10^0$$

$$300 \quad 80 \quad 6$$

101_2

$$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$$

$$1 \times 4 + 0 \times 2 + 1 \times 1$$

$$4 + 0 + 1 = 5$$

Binary/Decimal Represesntation of Integers

386_{10}

101_2

999_{10} $(1000) - 1$

$10^3 - 1$

Binary/Decimal Represesntation of Integers

386_{10} $0, 1, 2, \ldots, 9$

101_2

999_{10} $(1000) - 1$

$10^3 - 1$

$0, 1$

111_2

$1000 - 1$

$2^3 - 1 = 7$

Binary/Decimal Represesntation of Numbers

87.6_{10}

$$8 \times 10^1 + 7 \times 10^0 + 6 \times 10^{-1}$$

$$80 + 7 + 0.6$$

Binary/Decimal Represesntation of Numbers

87.6_{10} $\quad 8 \times 10^1 + 7 \times 10^0 + 6 \times 10^{-1}$

$80 + 7 + 0.6$

10.1_2

$1 \times 2^1 + 0 \times 2^0 + 1 \times 2^{-1}$

$2 + 0 + 1/2 = 2\,1/2$

Arithmetic

Compute $13 + 5$.

$$13_{10} = 1101_2$$

$$8\ 4\ 2\ 1$$

$$8 + 4 + 1 = 13$$

$$5_{10} = 101_2$$

$$4 + 1 = 5$$

Arithmetic

2

Compute 13 + 5.

$$\begin{array}{r} {\scriptstyle 1\ \ 1\ \ \ \ } \\ 1101 \\ 101 \\ \hline 10010 \end{array}$$

$16+2=18$

$13_{10} = 1101_2$

$5_{10} = 101_2$

$$\begin{array}{r} {\scriptstyle 1\ \ } \\ 94 \\ 7 \\ \hline 1 \end{array}$$

$4+7=11$

$1+1=10$

$$\begin{array}{r} {\scriptstyle 1\ \ } \\ 93 \\ 7 \\ \hline 100 \end{array}$$

Arithmetic

Compute 13 + 5.

$13_{10} = 1101_2$

$5_{10} = 101_2$

Compute $13 \cdot 5$

$$\begin{array}{r} 1101 \\ 101 \\ \hline 1101 \\ 1101 \\ \hline 1000001 \end{array} = 65_{10}$$

$5 \cdot 3$

$$\begin{array}{r} 101 \\ 11 \\ \hline 101 \\ 101 \\ \hline 1111 \end{array} = 15_{10}$$

Arithmetic

Compute $13 + 5$.

Compute $13 \cdot 5$

Compute $\cancel{13/5}$

$15/5$

$$\begin{array}{r} 11 \\ 101\,\overline{)\,1111} \\ 101 \\ \hline 101 \end{array}$$

$11 = 3_{10}$

General Fractions

Compute the binary representation of 1/3.

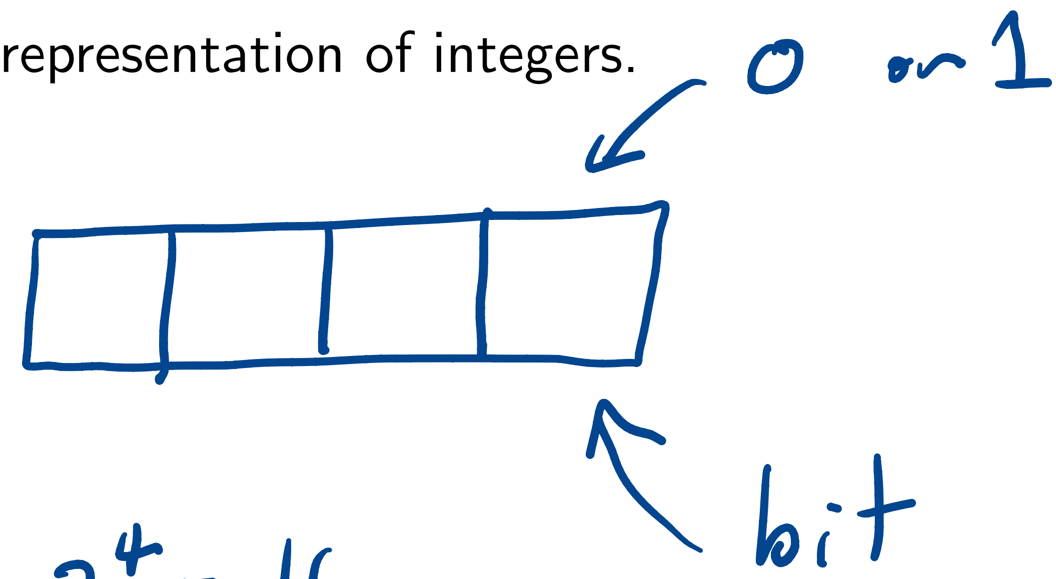
$$
\begin{array}{r|l}
 & 0.0101 \\
\hline
11 & 1.00000 \\
 & 11 \\
\cline{2-2}
 & 100
\end{array}
$$

$$\frac{1}{3} = 0.\overline{01}_2$$

$$\frac{1}{3} = 0.\overline{3}_{10}$$

Positive Integers On A Computer

E.g., 4-bit representation of integers.



$2^4 = 16$

different patterns.

$0000 \to 0$

$0001 \to 1$

$0010 \to 2$

$\vdots$

$1111 \to 15$

Positive Integers On A Computer

E.g., 4-bit representation of integers.

2^4 possibilities, $0 - 2^4 - 1$

$0 - 15$

In practice:

- 8-bits: 0 to $(2^8 - 1) = 255$
- 16-bits: 0 to $2^16 - 1 = 65535$
- 32-bits: 0 to $\sim 4 \times 10^9 = 4$ billion
- 64-bits: 0 to $\sim 10^{19}$.