Fixed Point Convergence

Math 426

University of Alaska Fairbanks

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Secant Method: Rate of Convegence

$f(x) = x^2 - 2, \; x_1 = 1, \; x_2 = 1.1$

$$\frac{|e_{k+1}|}{|e_k|^\alpha} \to C > 0$$

$$e_3 \approx 4 \times 10^{-1}$$
$$e_4 \approx 3 \times 10^{-1}$$
$$e_5 \approx 6 \times 10^{-2}$$
$$e_6 \approx 8 \times 10^{-3}$$
$$e_7 \approx 2 \times 10^{-4}$$
$$e_8 \approx 4 \times 10^{-7}$$
$$e_9 \approx 2 \times 10^{-11}$$
$$e_{10} \approx 4 \times 10^{-16}$$

$$|e_{k+1}| \approx C|e_k|^\alpha$$

bisection $\alpha = 1$

Newton: $\alpha = 2$

Secant Method: Rate of Convegence

Text:

Secant

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k e_{k-1}|} = C$$

Newton:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{\underbrace{|e_k|^2}} = C$$

$e_k \cdot e_{k-1}$

Secant Method: Rate of Convegence

Text:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k e_{k-1}|} = C$$

Newton:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = C$$

$$\frac{|e_{k+1}|}{|e_k|^\alpha} \to a$$

Suppose $e_{k+1} \sim a e_k^\alpha$.

Secant Method: Rate of Convegence

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$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k e_{k-1}|} = C$$

Newton:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = C$$

$$a^{-1} e_k \sim e_{k-1}^{\alpha}$$

$$a^{-1/\alpha} e_k^{1/\alpha} \sim e_{k-1}$$

Suppose $e_{k+1} \sim a e_k^{\alpha}$. $\quad e_k \sim a e_{k-1}^{\alpha} \Rightarrow$

$$\frac{|e_{k+1}|}{|e_k e_{k-1}|} \sim \frac{a|e_k|^{\alpha}}{|e_k||e_{k-1}|} \sim \frac{a|e_k|^{\alpha}}{a^{-1/\alpha}|e_k||e_k|^{1/\alpha}} \sim c|e_k|^{\alpha-1-1/\alpha}.$$

$$|e_k|\,|e_{k-1}|$$

$$\frac{|e_{k+1}|}{|e_k e_{k-1}|} \to C$$

$$e_k \to 0$$

$$\to c\,|e_k|^{\alpha-1-1/\alpha}$$

Secant Method: Rate of Convegence

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Newton:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = C$$

Suppose $e_{k+1} \sim a e_k^{\alpha}$.

$$\frac{|e_{k+1}|}{|e_k e_{k-1}|} \sim \frac{a|e_k|^{\alpha}}{|e_k||e_{k-1}} \sim \frac{a|e_k|^{\alpha}}{a^{-1/\alpha}|e_k||e_k|^{1/\alpha}} \sim c|e_k|^{\alpha-1-1/\alpha}.$$

$$e_k \to 0 \implies \alpha^2 - \alpha - 1 = 0$$

$\alpha - 1 - \frac{1}{\alpha} = 0$

$\alpha^2 - \alpha - 1 = 0$

Secant Method: Rate of Convegence

Text:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k e_{k-1}|} = C$$

Taylor's Theorem

Newton:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = C$$

$a \leq 0$

$e_{k+1} \sim a e_k^{\alpha}$

Suppose $|e_{k+1}| \sim a|e_k|^{\alpha}$.

$$\frac{|e_{k+1}|}{|e_k e_{k-1}|} \sim \frac{a|e_k|^{\alpha}}{|e_k||e_{k-1}|} \sim \frac{a|e_k|^{\alpha}}{a^{-1/\alpha}|e_k||e_k|^{1/\alpha}} \sim c|e_k|^{\alpha-1-1/\alpha}.$$

$$\frac{|e_{k+1}|}{|e_k|^{\alpha}} \to C$$

$$e_k \to 0 \implies \alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

$$\frac{1-\sqrt{5}}{2} < 0$$

$$e_k \to 0 \implies \alpha = \frac{1+\sqrt{5}}{2} \simeq 1.6$$

Convergence of Newton's Method

We showed that **if** Newton's method converges and $f'(x_*) \neq 0$ then

$$\frac{|e_{k+1}|}{|e_k|^2} \to \frac{f''(x_*)}{2f'(x_*)}. \qquad C$$

But we never showed that it would actually converge.

Convergence of Newton's Method

We showed that **if** Newton's method converges and $f'(x_*) \neq 0$ then

$$\frac{|e_{k+1}|}{|e_k|^2} \to \frac{f''(x_*)}{2f'(x_*)}.$$

But we never showed that it would actually converge.

Stragegy: formulate Newton's method as a fixed point method, determine conditions under which fixed point methods converge, and apply to Newton's method.

Fixed point methods

Want to solve

$$\Phi(x) = x.$$

$\Phi(x_k) = x_{k+1}$

Such a point is called a **fixed point** of Φ.

For Newton's method,

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

$\Phi(x) = x$

Fixed point methods

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$$\Phi(x) = x.$$

Such a point is called a **fixed point** of Φ.

For Newton's method,

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

If $\Phi(x_*) = x_*$,

$$x_* = x_* - \frac{f(x_*)}{f'(x_*)}$$

$$0 = -\frac{f(x_*)}{f'(x_*)}$$

so

$$f(x_*) = 0.$$

Fixed point methods

N_{max}

- $ftol$ $\quad |f(x_k)| < ftol$
- $xtol$ $\quad |x_k - x_{k-1}| < xtol$

Want to solve

$$\Phi(x) = x.$$

Such a point is called a **fixed point** of Φ.

For Newton's method,

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\Phi(x_*) = x_* - \frac{f(x_*)}{f'(x_*)}$$

If $\Phi(x_*) = x_*$,

$$x_* = x_* - \frac{f(x_*)}{f'(x_*)}$$

so

$$f(x_*) = 0.$$

If $f(x_*) = 0$ then $\Phi(x_*) = x_*$ (assuming $f'(x_*) \neq 0$)

$$= x_* - \frac{0}{f'(x_*)}$$

$$= x_*$$

Fixed point methods

Want to solve

$$\Phi(x) = x.$$

Such a point is called a **fixed point** of Φ.

For Newton's method,

$$\Phi(x) = x - \frac{f(x)}{f'(x)}$$

If $\Phi(x_*) = x_*$,

$$x_* = x_* - \frac{f(x_*)}{f'(x_*}$$

so

$$f(x_*) = 0.$$

If $f(x_*) = 0$ then $\Phi(x_*) = x_*$ (assuming $f'(x_*) \neq 0$.

The fixed points of Φ are exactly the roots of f (up to legalese).

Visualizing Fixed Point Iteration

$x = 1$

Suppose $\Phi(x) = x^2$. What are the fixed points? $x = 0$

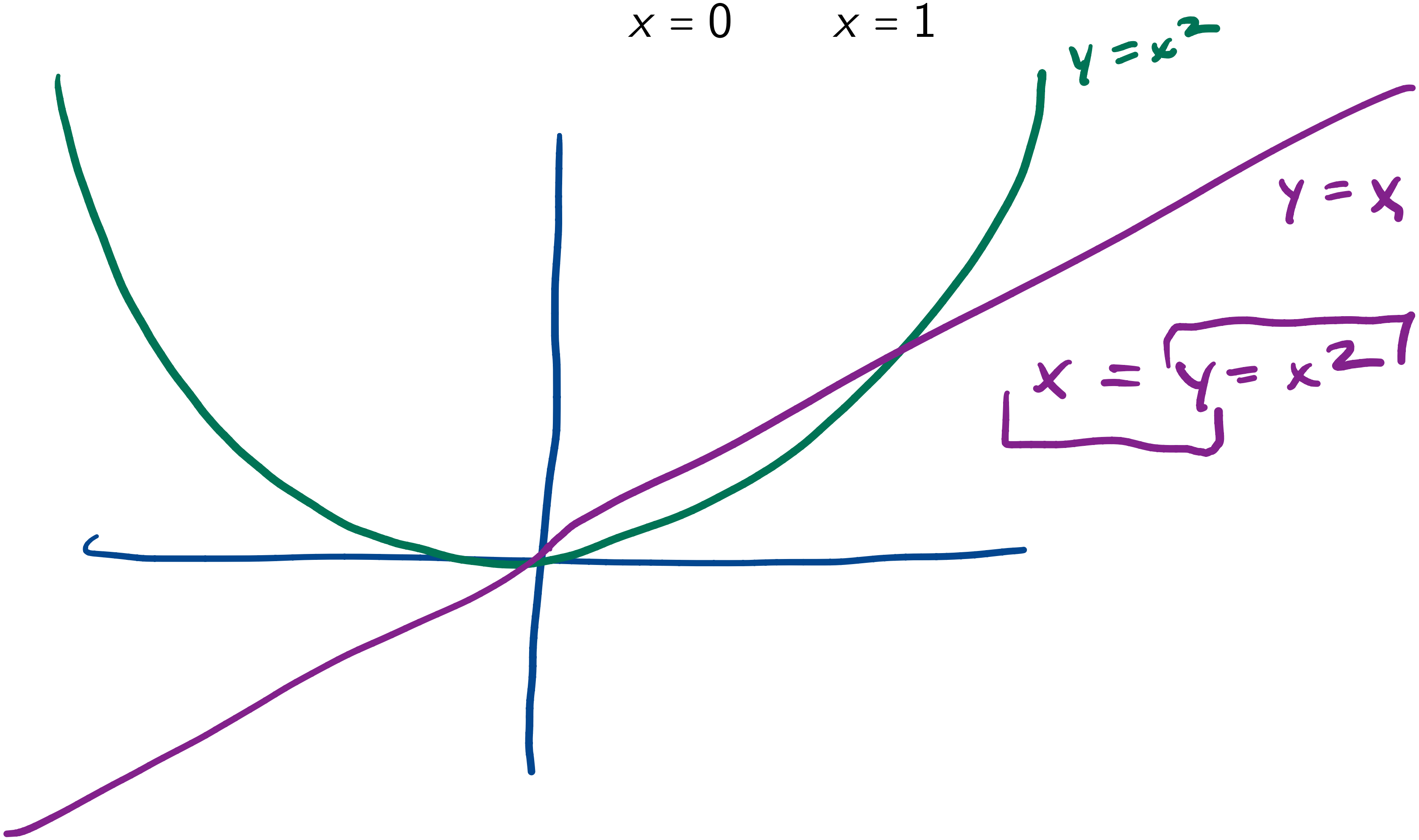
$\Phi(1) = 1^2 = 1$

$\Phi(0) = 0^2 = 0$

Visualizing Fixed Point Iteration

Suppose $\Phi(x) = x^2$. What are the fixed points?

$x = 0 \qquad x = 1$



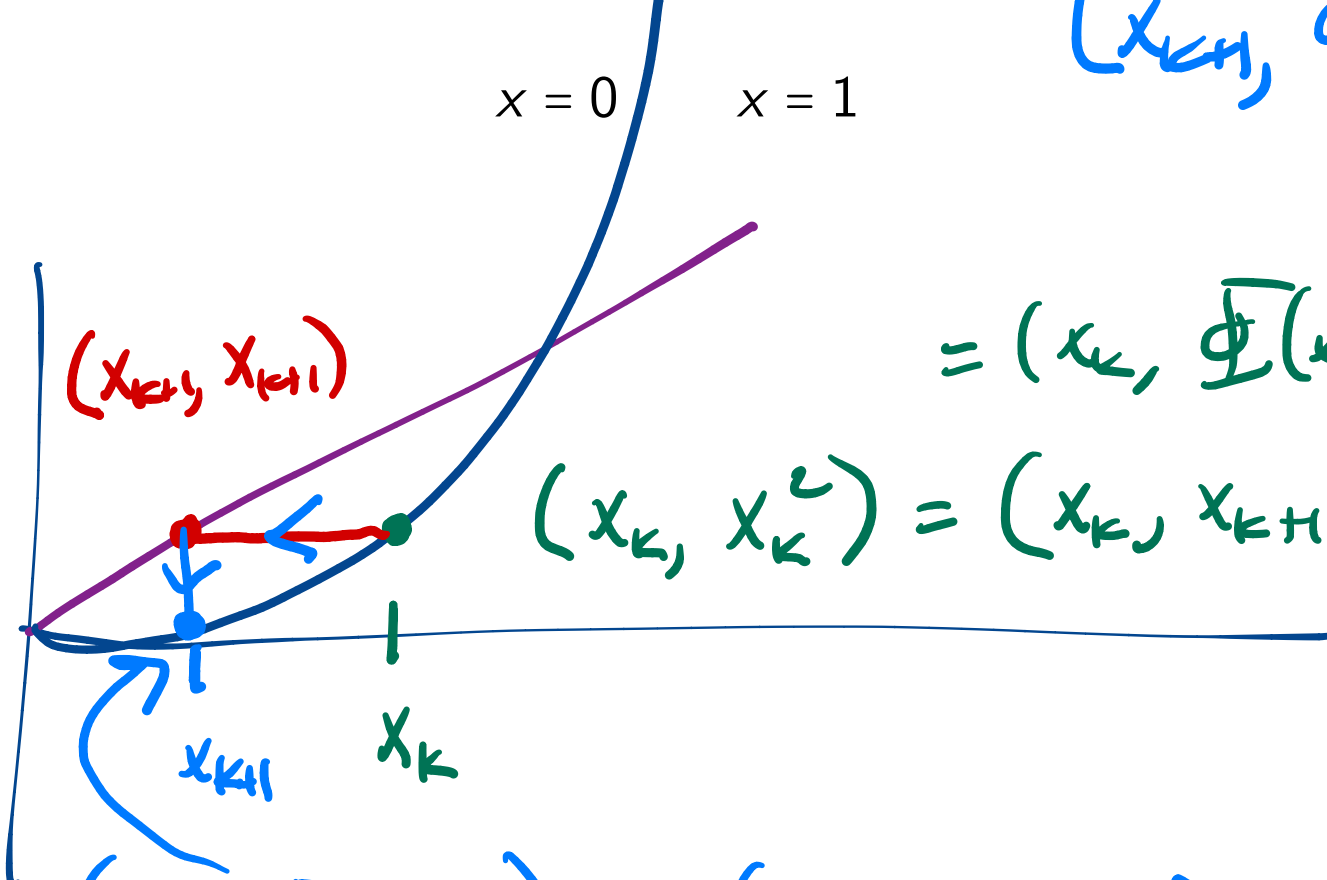
Visualizing Fixed Point Iteration

$(x_k, \phi(x_k))$

Suppose $\Phi(x) = x^2$. What are the fixed points?

$(x_{k+1}, \phi(x_{k+1}))$

$x = 0 \qquad x = 1$



$= (x_k, \Phi(x_k))$

$(x_k, x_k^2) = (x_k, x_{k+1})$

$(x_{k+1}, \Phi(x_{k+1})) = (x_{k+1}, x_{k+2})$

Visualizing Fixed Point Iteration

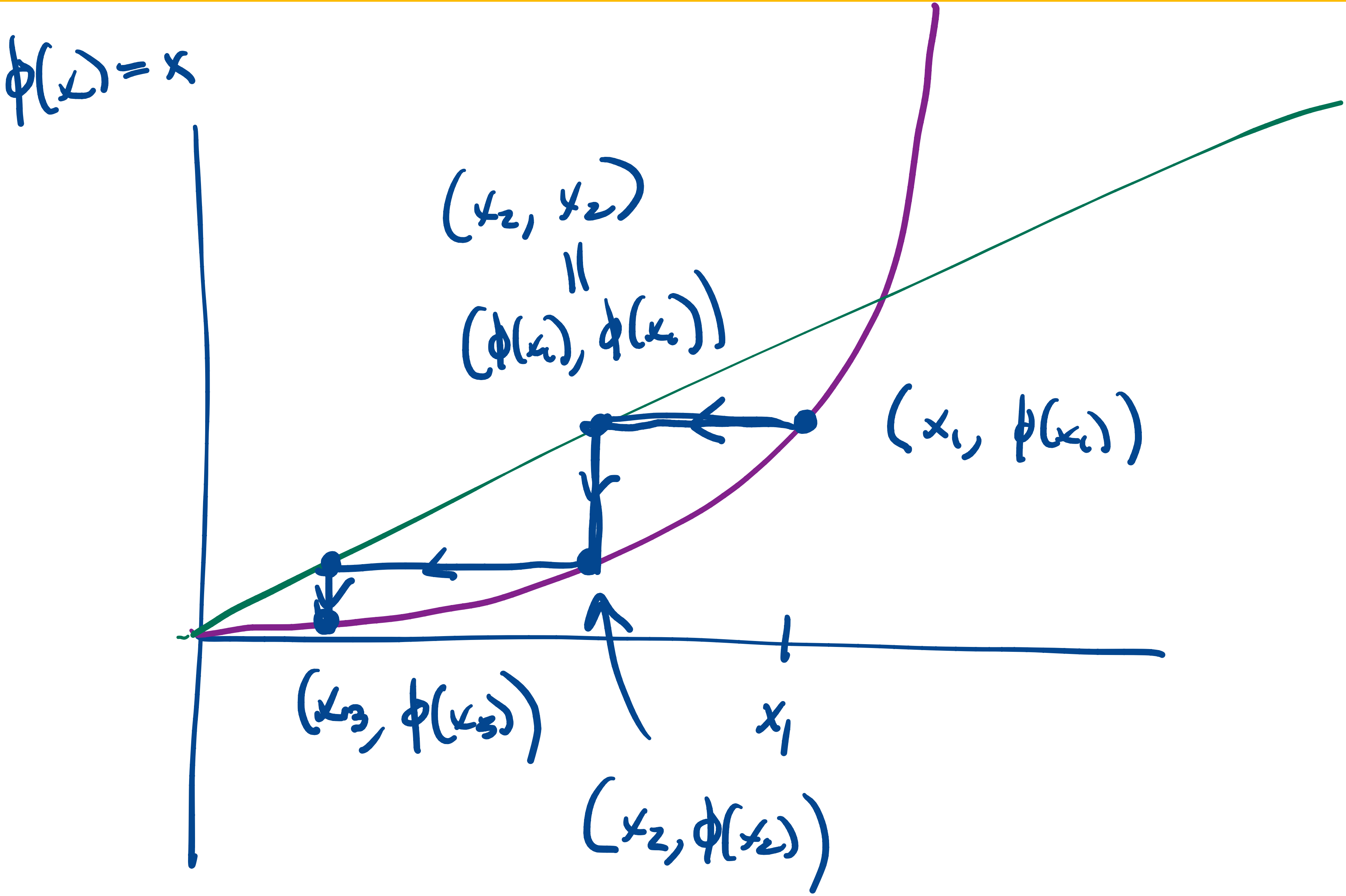
Suppose $\Phi(x) = x^2$. What are the fixed points?

$$x = 0 \qquad x = 1$$

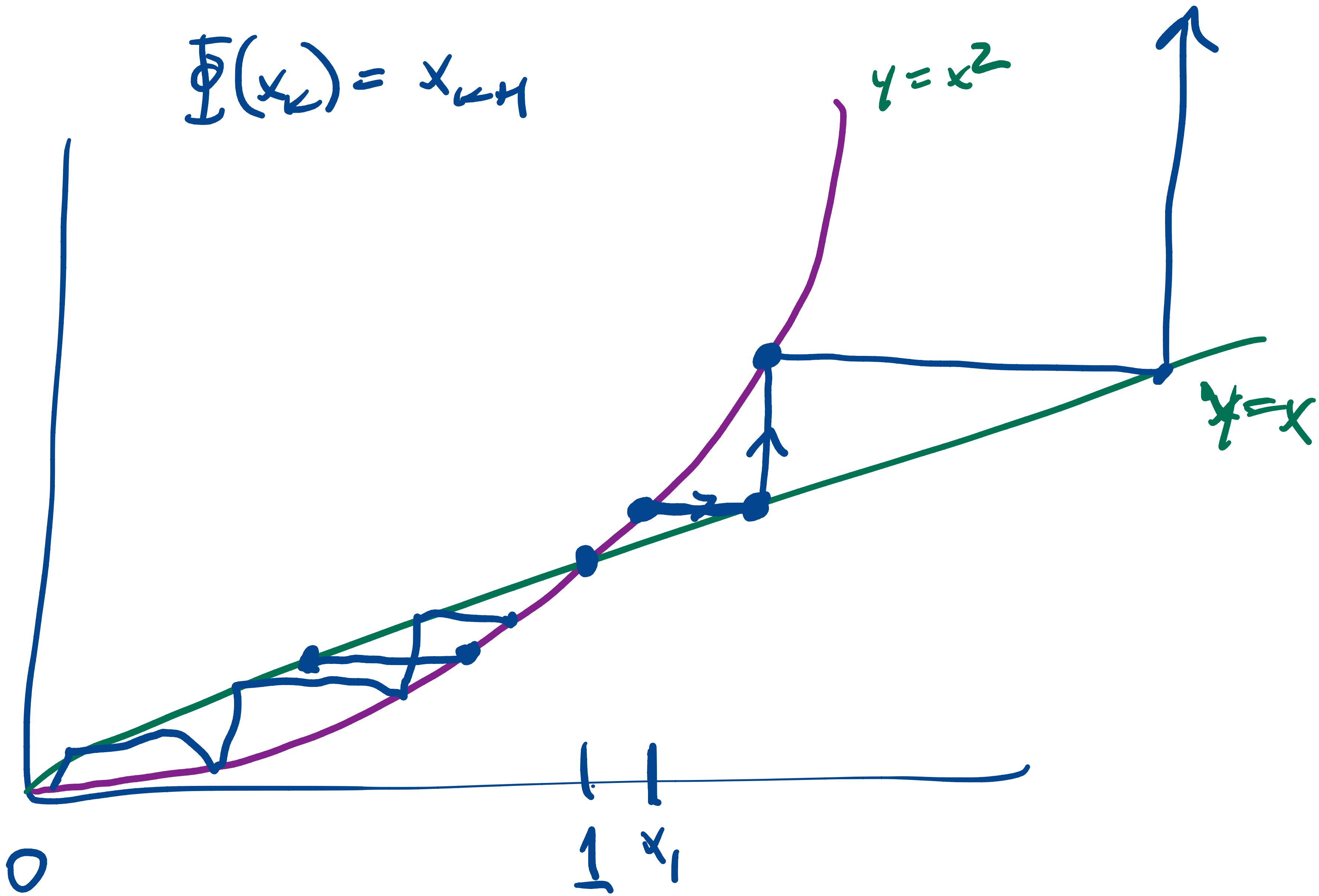
x_{k+2}

$$(x_k, \Phi(x_k)) \to (\Phi(x_k), \Phi(x_k)) = (x_{k+1}, x_{k+1}) \to (x_{k+1}, \Phi(x_{k+1}))$$

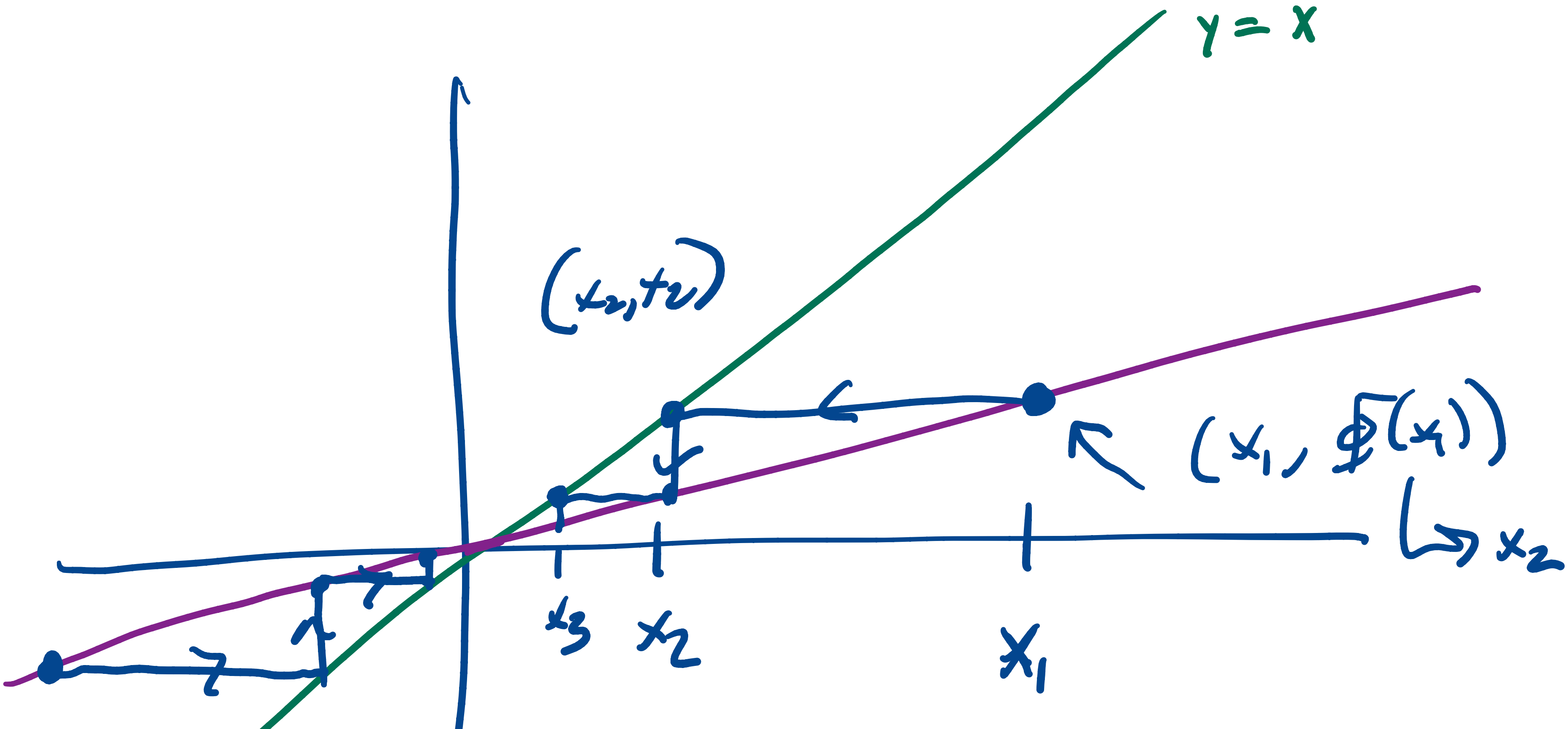
Two cases for $\Phi(x) = x^2$



Two cases for $\Phi(x) = x^2$



$\Phi(x) = x/2$



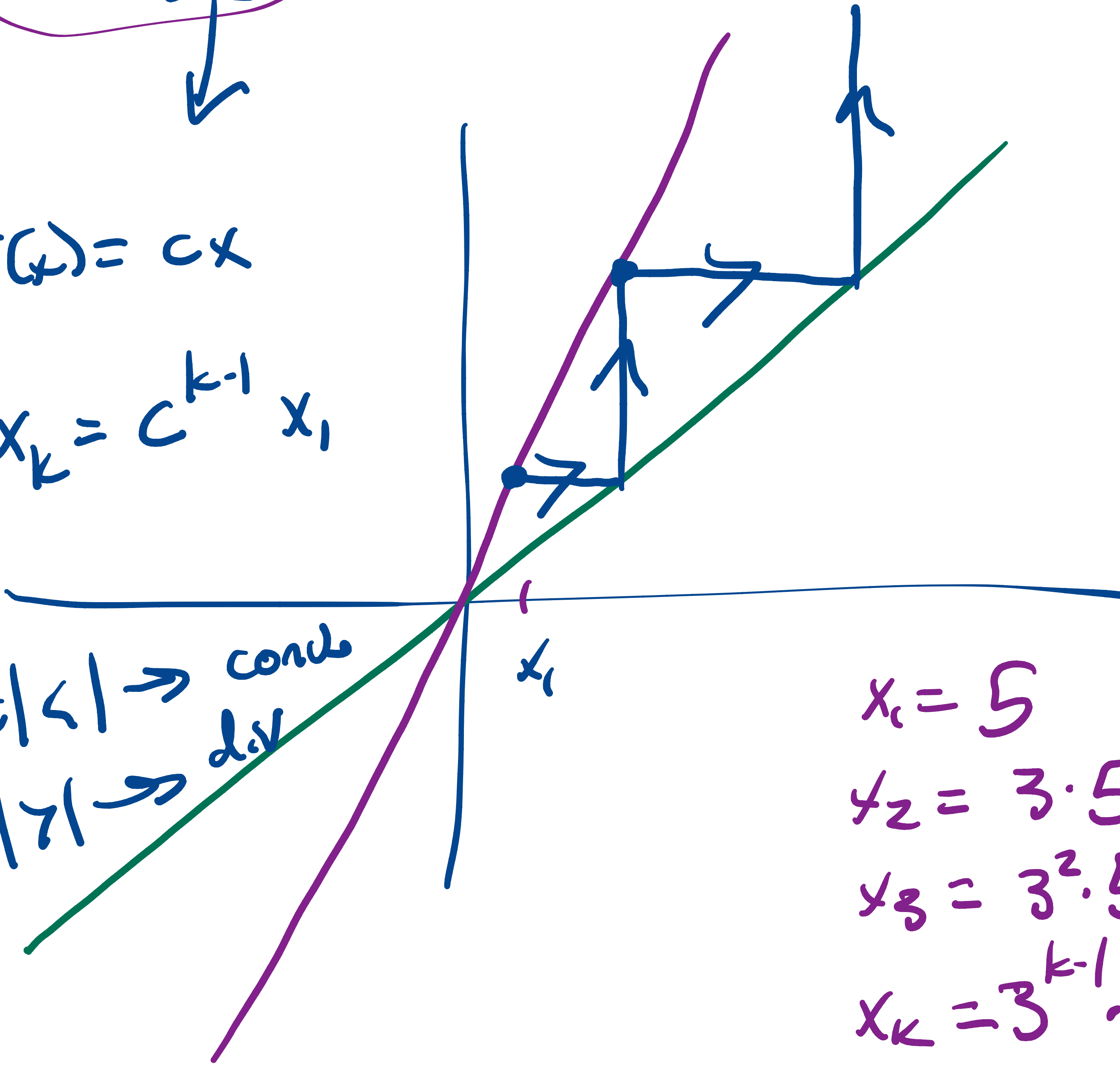
$$x_2 = \Phi(x_1) = \frac{1}{2} x_1$$

$$x_3 = \Phi(x_2) = \frac{1}{2} x_2 = \frac{1}{4} x_1$$

$$\Phi(x) = 3x$$

$$\Phi(x) = cx$$

$$x_k = c^{k-1}\, x_1$$



$|c| < 1 \rightarrow$ conv

$|c| > 1 \rightarrow$ div

$x_1 = 5$

$x_2 = 3 \cdot 5$

$x_3 = 3^2 \cdot 5$

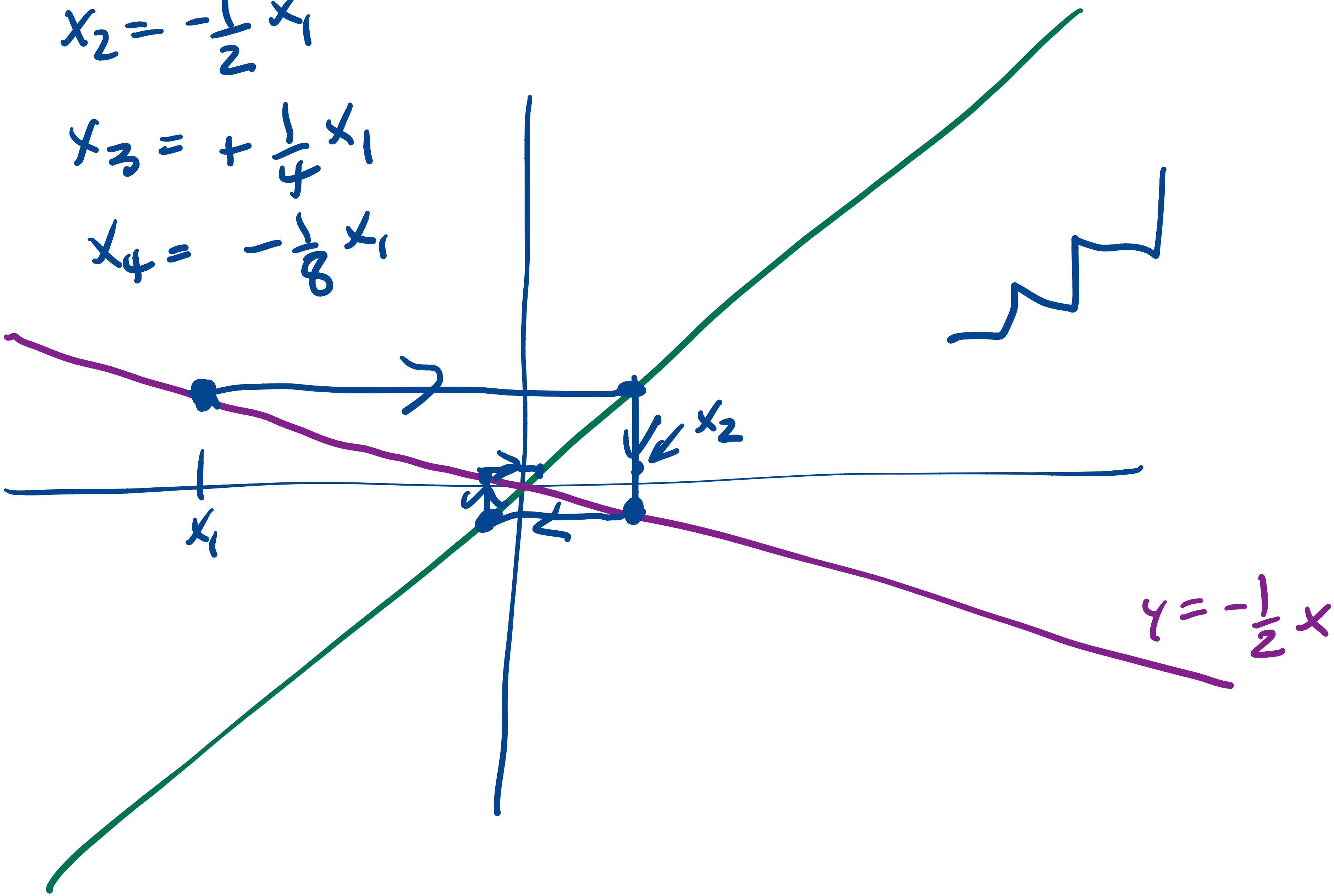
$x_k = 3^{k-1} \cdot 5$

$\Phi(x) = -x/2$

$x_2 = -\frac{1}{2} x_1$

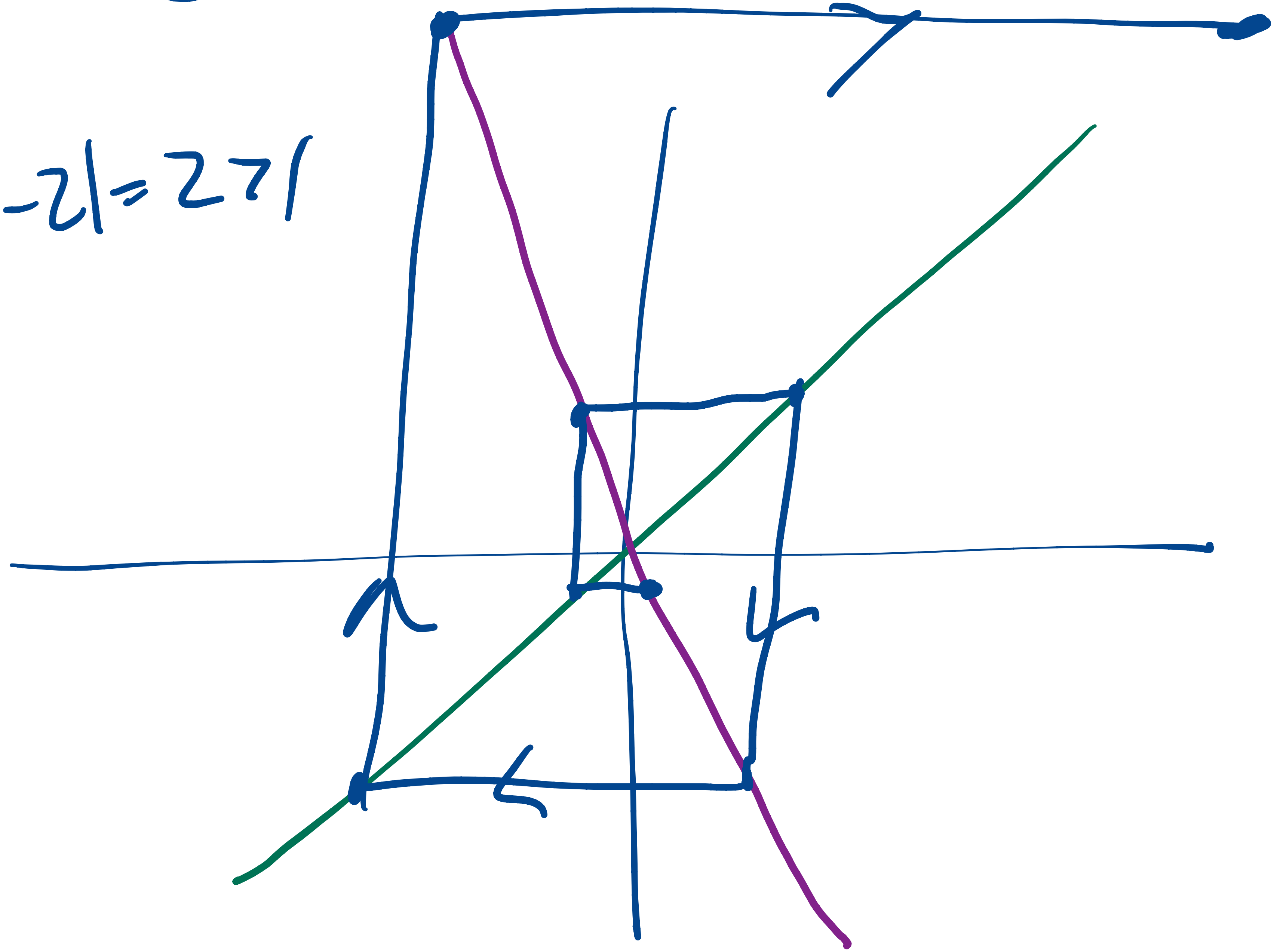
$x_3 = +\frac{1}{4} x_1$

$x_4 = -\frac{1}{8} x_1$



$\Phi(x) = -2x$

$|-2| = 2 > 1$



$\Phi(x) = -2x$