Secant Method

Math 426

University of Alaska Fairbanks

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When do you stop?

① After a given number of iterations.
(Error) N_{max}

② $|f(x_k)| < \varepsilon_{ftol}$

$f(x_*) = 0$

④
$$\frac{|x_{k+1} - x_k|}{|x_{k+1}|} < \varepsilon_{rtol}$$

relative tolerance

③ $|x_{k+1} - x_k| < \varepsilon_{atol}$

x_*

$\hookrightarrow$ absolute.

Convergence Rates

$$f(x_*) = 0$$

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for some positive constant C.

$|e_{k+1}| \approx C|e_k|^\alpha$, so bigger α is better

For large values of k

$$|e_{k+1}| \approx C|e_k|^\alpha$$

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$$\frac{e_{k+1}}{e_k} \sim \frac{1}{2}$$

Bisection: linear ($\alpha = 1$) $e_{k+1} \sim e_k/2$

Newton's method: quadratic ($\alpha = 2$) [Mostly!]

$$\frac{|e_{k+1}|}{|e_k|^2} \to C$$

Convergence Theorem

Theorem

Suppose $f \in C^2(\mathbb{R})$ *and* $f(x_*) = 0$. **If** $f'(x_*) \neq 0$ *then there is an* $\epsilon > 0$ *such that if* $x_1 \in (x_* - \epsilon, x_* + \epsilon)$ *then*

1. $x_k \to x_*$
2. $\dfrac{|e_{k+1}|}{|e_k|^2} \to \left|\dfrac{f''(x_*)}{2f'(x_*)}\right|$

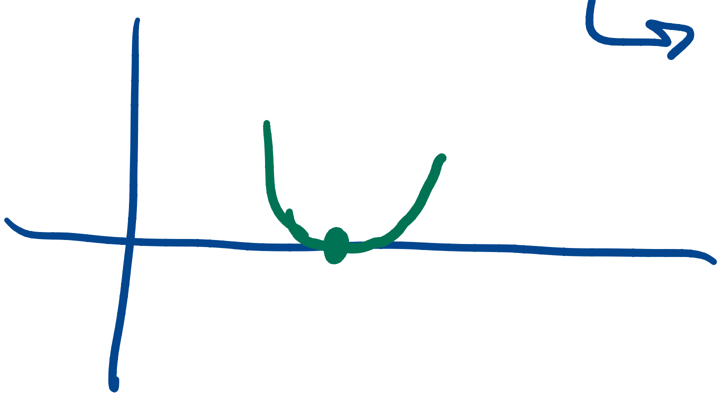
$\to 2f'(x_*) = 0$

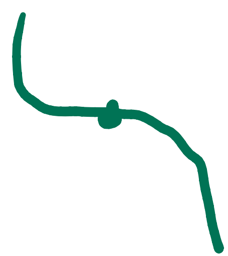
$f(x_*) = 0$

$f'(x_*) = 0$

$f(x) = x^2$

$f(x) = x^3$





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Consider $f(x) = x^2$, $x_* = 0$.

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$$|e_{k+1}| = \frac{1}{2}|e_k|$$

Iteration function

$$\Phi(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2}{2x} = \frac{x}{2}.$$

Iterates:

$$x_{k+1} = \frac{x_k}{2}$$

$$e_{k+1} = x_* - x_{k+1} = x_* - \frac{x_k}{2}$$

$$e_{k+1} = 0 - x_{k+1} = 0 - \frac{x_k}{2}$$

$$|e_{k+1}| = \left|\frac{x_k}{2}\right| = \left|\frac{-e_k}{2}\right| = \frac{1}{2}|e_k|$$

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Since $x_* = 0$,

$$e_k = x_* - x_k = -x_k$$

and

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The order of convergence is **linear**. Fortunately, this is rare.

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$1 - \frac{1}{(n+1)}$

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$|e_{k+1}| \sim \left(1 - \frac{1}{n+1}\right) |e_k|$

and

$$\frac{|e_{k+1}|}{|e_k|} = \frac{1}{2}.$$

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$=0$

See text: if $f(x_*), f'(x_*) = 0, \ldots, f^{(n)}(x_*) = 0$ and $f^{(n+1)}(x_*) \neq 0$ then linear convegence, but $C = 1 - 1/(n+1)$

Quasi Newton Methods

Newton's method applies to much larger systems than one scalar function of one real variable. Computing the derivative for large systems turns out to be both expensive and error-prone to code. If you get it wrong, convergence rate goes back to linear.

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Newton's method applies to much larger systems than one scalar function of one real variable. Computing the derivative for large systems turns out to be both expensive and error-prone to code. If you get it wrong, convergence rate goes back to linear. Strategy for a quasi-newton method:

$$x_{k+1} = x_k - \frac{f(x_k)}{m(x_k)}$$

where $m(x_k)$ is an approximation of $f'(x_k)$.

Constant slope

Just use $m(x_k) = f'(x_1) =: m$ always. This is a pretty crappy idea.

$$x_{k+1} = x_k - \frac{f(x_k)}{m}$$

$$e_{k+1} = e_k + \frac{f(x_k)}{m}$$

$$\underbrace{x_{k+1} - x_*}_{-e_{k+1}} = \underbrace{x_k - x_*}_{-e_k} - \frac{f(x_k)}{m}$$

Taylor:

$$f(x_k) = f(x_*) + f'(x_*)\overbrace{(x_k - x_*)}^{-e_k} + \frac{f''(\xi)}{2}(x_k - x_*)^2$$

$$= f'(x_*)e_k + O(e_k^2)$$

$-$

Not $\neq 0$ generally.

$$e_{k+1} = \left[1 - \frac{f'(x_*)}{m}\right] e_k + O(e_k^2)$$

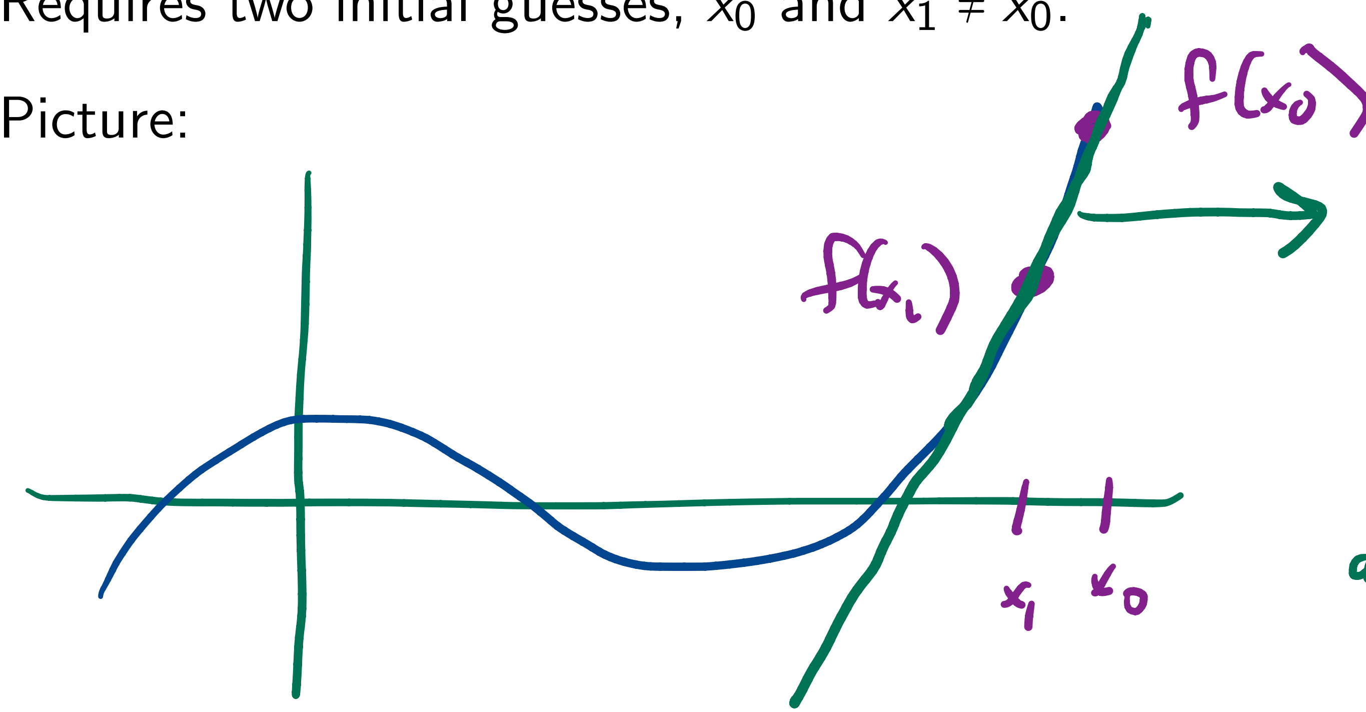
Linear convergence.

$m = f'(x_*)$

Secant Method

Requires two initial guesses, x_0 and $x_1 \neq x_0$.

Picture:



$$\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

approximate slope

Secant Method

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Picture:

$$\underset{1}{m_k} = \frac{f(\overset{1}{x_k}) - f(\overbrace{x_{k-1}}^{0})}{x_k - x_{k-1}}$$

$$x_{k+1} = \overset{1}{x_k} - \frac{\overbrace{f(x_k)}^{0}}{m_k}.$$

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Secant Method: Convegence

$$x_{k+1} = x_k - \frac{f(x_k)}{m_k}$$

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Secant Method: Convegence

$$x_{k+1} = x_k - \frac{f(x_k)}{m_k}$$

$$m_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$f(x_k) = f'(\theta_k)e_k$$

$$f(x_k) = -f(\theta_k)e_k$$

$$f(x_k) - f(x_{k-1}) = f'(\xi_k)e_k$$

Taylor's theorem centered at $x_* = a$

$$f(x_k) = \underbrace{f(x_*)}_{0} + f'(\theta_k)\overbrace{(x_k - x_*)}^{-e_k}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots$$

$$+ \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

$$+ \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}$$

$$f(x) = f(a) + f'(\xi)(x-a)$$

between x and a

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(\xi)(x-a)^2$$

Secant Method: Convegence

$$x_{k+1} = x_k - \frac{f(x_k)}{m_k}$$

$$m_k = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} =$$

$$f(x_k) = f'(\theta_k) e_k$$

Taylor's Thm

$$f(x_k) - f(x_{k-1}) = f'(\xi_k) \cancel{e_k} (x_k - x_{k-1})$$

$$e_{k+1} = \underbrace{\left[1 - \frac{f'(\theta_k)}{f'(\xi_k)}\right]}_{\hookrightarrow \text{ if this is } < 1} e_k$$

$f'(\theta_k) \approx f'(x_*)$

$f'(\xi_k) \approx f'(x_*)$

At least linear convergence.

Secant Method: Rate of Convegence

$$f(x) = x^2 - 2,\ \underline{x_1 = 1,\ x_2 = 1.1}$$

$$e_3 \approx 4 \times 10^{-1}$$

$$e_4 \approx 3 \times 10^{-1}$$

$$e_5 \approx 6 \times 10^{-2}$$

$$e_6 \approx 8 \times 10^{-3}$$

$$e_7 \approx 2 \times 10^{-4}$$

$$e_8 \approx 4 \times 10^{-7}$$

$$e_9 \approx 2 \times 10^{-11}$$

$$e_{10} \approx 4 \times 10^{-16}$$