# Newton's Method (III)

Math 426

University of Alaska Fairbanks

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# What's a Taylor polynomial?

$f(x)$ $\qquad$ $x = a$

$k^{th}$ order Taylor polynomial.

$\hookrightarrow$ $P(x)$ $\qquad$ $k^{th}$ order polynomial.

$$P(a) = f(a)$$
$$P'(a) = f'(a)$$
$$P''(a) = f''(a)$$
$$\vdots$$
$$P^{(k)}(a) = f^{(k)}(a)$$

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

## Newton's Method Recap

Want to solve

$$f(x) = 0.$$

With a guess $x_1$ for the root location we instead solve where the linearization centered at $x_1$ has a root.

$x_1 = a$

$$L(x) = f(x_1) + f'(x_1)(x - x_1)$$

$$L(x) = 0 \implies x = x_1 - \frac{f(x_1}{f'(x_1)}.$$

Iterates:

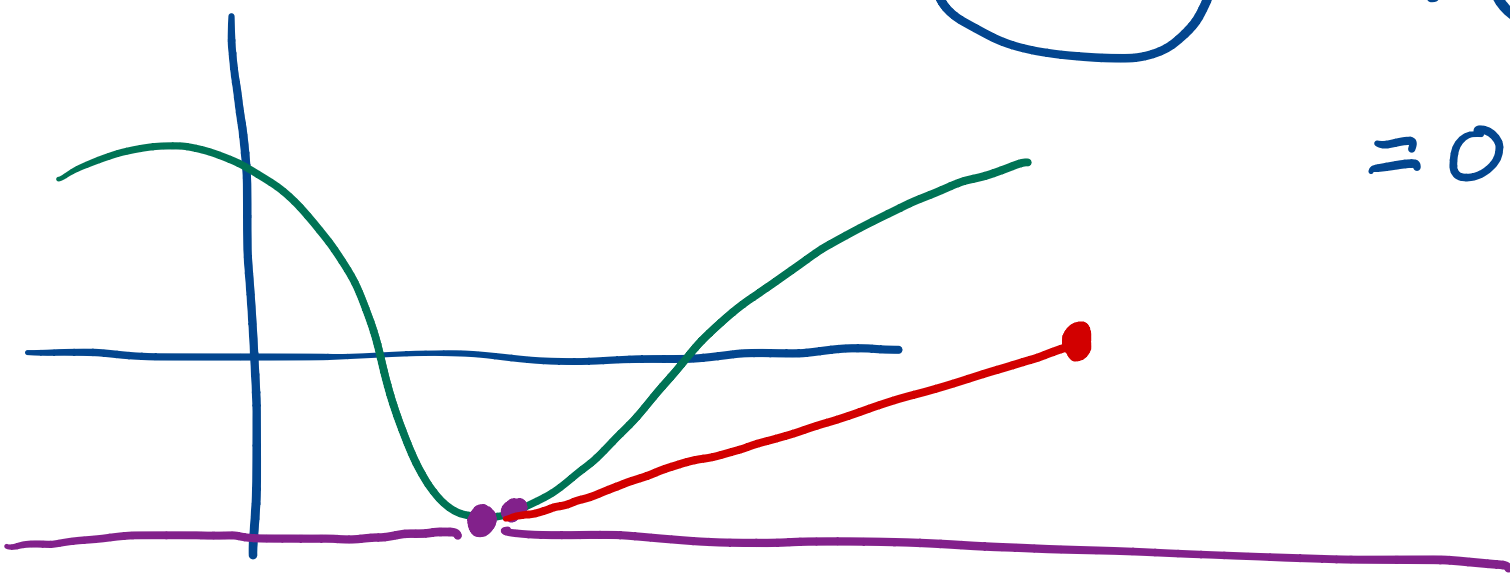
iteration function

$$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f(x_k)}}_{\Phi(x_k)}$$

## What could possibly go wrong?

$$f(x_{k+1}) = x_k - \frac{f(x_k)}{f(x_{k+1})} \longrightarrow f'(x_k)$$

$= 0?$



# Application to $\sqrt{2}$

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$x_k^2 - 2$

$2x_k$

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$$x_{k+1} = x_k - \frac{x_k}{2} + \frac{2}{2x_k} = \frac{x_k}{2} + \frac{1}{x_k}.$$

$\Phi(x_k)$

$x_{k+1} = \Phi(x_k)$

$x_k \to x_*$ we hope!

$f(x_*) = 0$

# Size of errors:

$e_3 \cdot e_3 \sim 10^{-6}$

$e_4 \cdot e_4 \sim 10^{-12}$

$$e_1 = 0.4 \approx 10^{-1}$$

$$e_2 = 0.08 \approx 10^{-1}$$

$$e_3 = 0.002 \approx 10^{-3}$$

$2 \cdot 3$

$$e_4 = 0.000002 \approx 10^{-6}$$

$2 \cdot 6$

$$e_5 = 1.6 \times 10^{-12} \approx 10^{-12}$$

$10^{-24}$

$2 \cdot 12$

Pattern?

$e_{k+1} \sim e_k \cdot e_k$

$e_{k+1} \sim \frac{1}{2} e_k$

# Quadratic convergence

root $f(x_*) = 0$

$$e_k = x_* - x_k$$

$$|e_{k+1}| \approx C|e_k|^2$$

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^2} = C$$

For large $\cancel{C}$ $k$,

$$|e_{k+1}| \approx Ce_k^2 \qquad C|e_k|^2$$

If $e_k$ is small, and $C$ doesn't get in the way, then $e_{k+1}$ is much smaller than $e_k$.

# Convergence of Newton's Method

twice continuously differentiable

## Theorem

*Suppose* $f \in C^2(\mathbb{R})$ *and* $f(x_*) = 0$ . *If* $f'(x_*) \neq 0$ *then there is an* $\epsilon > 0$ *such that if* $\underbrace{x_1 \in (x_* - \epsilon, x_* + \epsilon)}_{\text{if we start close enough to } x_*}$ *then* .

1. $x_k \to x_*$ .
2. $\dfrac{|e_{k+1}|}{|e_k|^2} \to \left|\dfrac{f''(x_*)}{2f'(x_*)}\right|$

$$f(x) = x^2 - 2$$
$$f'(x) = 2x$$
$$f''(x) = 2$$

$$\frac{f''(\sqrt{2})}{2f'(\sqrt{2})}$$

$$\frac{2}{2 \cdot 2 \cdot \sqrt{2}}$$

$$\frac{1}{2\sqrt{2}} \sim \frac{1}{3}$$

# Motivation for Convergence Rate

$x = x_*$ $\quad a = x_k$ $\quad \xi$ is between $x_*$ and $x_k$

$$f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\xi)(x_* - x_k)^2$$

# Motivation for Convergence Rate

solve for $x_*$

$=0$ $\quad$ $e_k$

$$0 = f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\xi)(x_* - x_k)^2$$

$$x_* = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{x_{k+1}} + \frac{1}{2} \frac{f''(\xi)}{f'(x_k)} e_k^2$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$\boxed{x_* - x_{k+1}} = e_{k+1}$$

# Motivation for Convergence Rate

$$f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\xi)(x_* - x_k)^2$$

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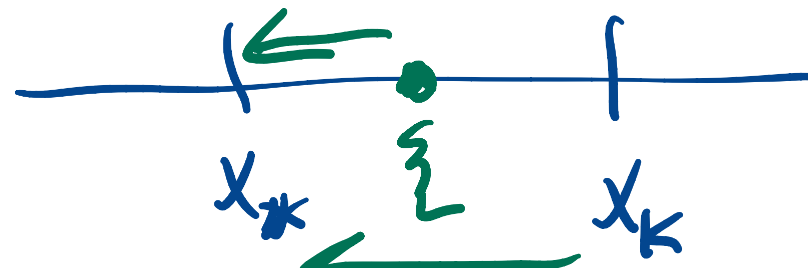



$$f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\xi)(x_* - x_k)^2$$

$$x_* = x_k - \frac{f(x_k)}{f'(x_k)} + \frac{1}{2}\frac{f''(\xi)}{f'(x_k)} e_k^2$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$e_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(x_k)} e_k^2.$$

$$\frac{e_{k+1}}{e_k^2} = \frac{1}{2}\frac{f''(\xi)}{f'(x_k)}$$

$$\to \frac{1}{2}\frac{f''(x_*)}{f'(x_*)} \neq 0$$

If $x_k \to x_*$

$f'(x_k) \to f'(x_*)$

$\xi$ is between $x_*$ and $x_k$

$$f''(\xi) \to f''(x_*)$$

# Two Caveats

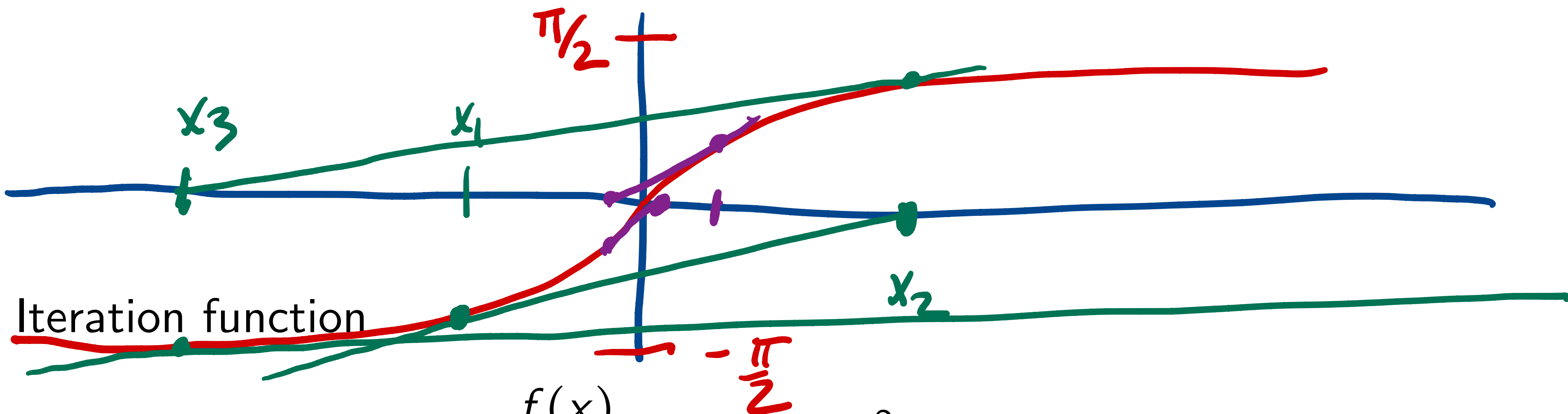
To get convergence, need

1. $f'(x_*) \neq 0$
2. $x_1 \in (x_* - \epsilon, x_* + \epsilon)$ for some $\epsilon$ you don't know.

# Need to start 'near' a root

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

$$f(x) = \arctan(x)$$




Iteration function

$$\Phi(x) = x - \frac{f(x)}{f'(x)} = x - (1 + x^2)\arctan(x)$$

MATLAB Demo: $x_1 = 1$ and $x_1 = 2$