Partial Pivoting

Math 426

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Why we pivot

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \mathbf{b} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$=$

Want to solve $A\mathbf{x} = \mathbf{b}$.

Find the LU factorization of

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Row exchange by permutation

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

permutation matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$PA = \begin{bmatrix} (0,1)A \\ (1,0)A \end{bmatrix} = \begin{bmatrix} \text{row 2 of } A \\ \text{row 1 of } A \end{bmatrix}$$

Row exchange by permutation

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Bigg(\underbrace{A \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{2nd col of } A}, \underbrace{A \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{1st col of } A} \Bigg) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Transform the problem

$$A\mathbf{x} = \mathbf{b}$$

$$PA\mathbf{x} = P\mathbf{b}$$

Now do LU factorization:

$$PA = LU$$

(P, L, U)

Permutation Matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

All entries are zero except

- Each row has exactly one 1.
- Each column has exactly one 1.

$PA =$

AP

E.g.

$$P\ \cancel{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow (2, 1, 3)$$

Multiply PA to permute rows of A. Multiply AP to permute columns of A. $\underbrace{P^2 = I.}$

$E^2 = I$

This P is special.

$E \to$ looks like I with two rows interchanged.

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E^2 = I$$

$\hookrightarrow$ swap rows 1 & 2.

Full LU Factorization

$$PA = LU$$

To solve

$$A\mathbf{x} = \mathbf{b}.$$

1. Observe this equation is equivalent to $PA\mathbf{x} = P\mathbf{b}$.
2. So $LU\mathbf{x} = P\mathbf{b}$
3. Solve $L\mathbf{b}' = P\mathbf{b}$.
4. Solve $U\mathbf{x}\mathbf{b}'$.

Pivoting even without zeros

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \qquad Ax=b \qquad x=\begin{pmatrix}1\\1\end{pmatrix} \qquad \hat{A}\hat{x}=b$$

$$\hat{A} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A \qquad \begin{pmatrix} A_{11} \\ A_{21} \\ \vdots \\ A_{n1} \end{pmatrix}$$

Solution:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1+a \\ 1-a \end{pmatrix}$$

$$a = \frac{10^{-20}}{1-10^{-20}} \approx 10^{-20}$$

Partial Pivoting

Pivoting even without zeros

$$\hat{A}\hat{x} = b$$

Using IEEE doubles:

$$\hat{A} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1-10^{20} \end{bmatrix}$$

$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\hat{A}\hat{x} = b$$

$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$Ly = b$$

$$\hat{U}x = y$$

$$\begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y_1 = 1$$

$$y_2 = 2 - 10^{20}$$

$$\hat{y}_2 = -10^{20}$$

$$\hat{y} = \begin{bmatrix} 1 \\ -10^{20} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} 1 \\ -10^{20} \end{bmatrix}$$

$$\begin{bmatrix} 1+a \\ 1-a \end{bmatrix}$$

$$a \approx 10^{-20}$$

$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L\hat{y} = \underset{\approx}{\underset{\uparrow}{\tilde{b}}}$$

$$\hat{U}\hat{x} = \hat{y}$$

$$\begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10^{20} \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\tilde{U} \quad \begin{pmatrix} \nabla & * \\ 0 & \end{pmatrix}$$

P

$$E_3 \, E_2 \, E_1$$

P

$$P = [1, \ldots\ldots, n] \longrightarrow \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$$