


Partial Pivoting

Math 426

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Why we pivot

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$


Want to solve $A\mathbf{x} = \mathbf{b}$.

Find the LU factorization of

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Row exchange by permutation

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{permutation matrix}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$PA = \begin{bmatrix} (0,1)A \\ (1,0)A \end{bmatrix} = \begin{bmatrix} \text{row 2 of } A \\ \text{row 1 of } A \end{bmatrix}$$

Row exchange by permutation

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\underbrace{A \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\substack{\text{2nd col} \\ \text{of } A}}, \underbrace{A \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\substack{\text{1st col} \\ \text{of } A}} \right) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Transform the problem

$$A\mathbf{x} = \mathbf{b}$$

$$PA\mathbf{x} = P\mathbf{b}$$

Now do LU factorization:

$$PA = LU$$

(P, L, U)



Permutation Matrices

All entries are zero except

- ▶ Each row has exactly one 1.
- ▶ Each column has exactly one 1.

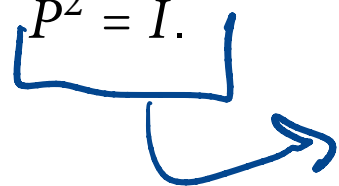
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} PA = \begin{matrix} \downarrow \\ \downarrow \\ \downarrow \end{matrix}$$

AP

E.g.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow (2, 1, 3)$$

Multiply PA to permute rows of A . Multiply AP to permute columns of A . $P^2 = I$.



This P is special.

$E \rightarrow$ looks like I
with two rows interchanged.

$$E^2 = I$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E^2 = I$$

→ swap rows 1 & 2.

Full LU Factorization

$$PA = LU$$

To solve

$$Ax = \mathbf{b}.$$

1. Observe this equation is equivalent to $PAx = P\mathbf{b}.$
2. So $LU\mathbf{x} = P\mathbf{b}$
3. Solve $L\mathbf{b}' = P\mathbf{b}.$
4. Solve $U\mathbf{x} = \mathbf{b}'.$

Pivoting even without zeros

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_A$$

$$Ax = b$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\hat{A}\hat{x} = b$$

$$\hat{A} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}; \mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Solution:

$$\hat{\mathbf{x}} = \begin{pmatrix} 1 + a \\ 1 - a \end{pmatrix}$$

$$a = \frac{10^{-20}}{1 - 10^{-20}} \approx 10^{-20}$$

$$\left(\begin{array}{c|c} A_{11} & \\ \hline A_{21} & \\ \vdots & \\ A_{n1} & \end{array} \right) \leftarrow A$$

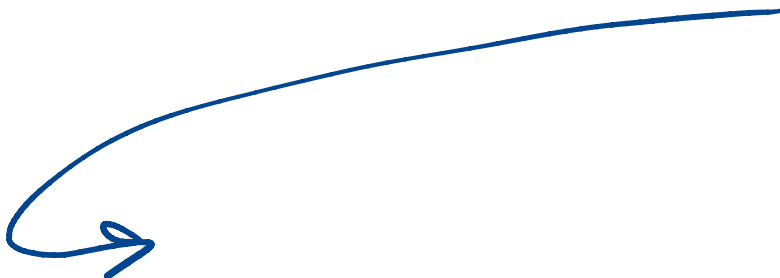
Partial Pivoting

Pivoting even without zeros

$$\hat{A}\hat{x} = \hat{b}$$

Using IEEE doubles:

$$\hat{A} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}; \hat{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$


$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$
$$U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1 - 10^{20} \end{bmatrix}$$
$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$

$$b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\hat{A} \hat{x} = b$$

$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L y = b$$

$$\hat{U} x = y$$

$$\begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y_1 = 1$$

$$y_2 = 2 - 10^{20}$$

$$\hat{y}_2 = -10^{20}$$

$$\hat{y} = \begin{bmatrix} 1 \\ -10^{20} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 1 \\ -10^{20} \end{bmatrix}$$

$$\begin{bmatrix} 1+a \\ 1-a \end{bmatrix}$$

$$a \approx 10^{-20}$$

$$\hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$L \hat{y} \approx b$$

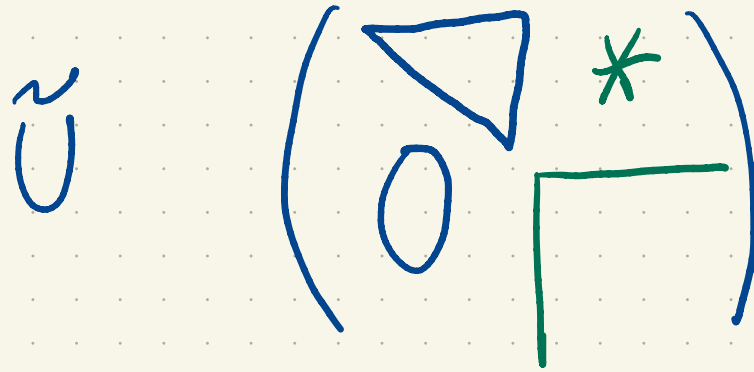
$$\hat{U} \hat{x} = \hat{y}$$

$$\begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -10^{20} \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



P

