Computational Complexity

Math 426

University of Alaska Fairbanks

October 5, 2020

Counting FLOPs

We'll count the number of floating point operations done by a computation (+,-,·,/).

E.g: Take the dot product of two n-dimensional vectors:

$$x_1 y_1 + \cdots + x_n y_n$$

$\underline{n}$ mults, $\underline{n-1}$ additions $2n-1$ operations

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E.g.: Multiply an $n \times n$ matrix A by an n-dimensional vector $\mathbf{x}$.

dot product of

$$\underbrace{\vec{r}_i \cdot \vec{x}}$$

$$A\mathbf{x} = \begin{pmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_n^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{r}_1^T \mathbf{x} \\ \vdots \\ \mathbf{r}_n^T \mathbf{x} \end{pmatrix}$$

$2n-1$

$n(2n-1) = 2n^2 - n$

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Foreshadowing: In $2n^2 - n$, does anybody care about the $-n$ when n is large?

20000 − 100 2000000 − 1000

Lower triangular solve

L lower triangular, all 1's on diagonal.

$$L\mathbf{c} = \mathbf{b}$$

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$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ L_{21} & 1 & 0 & \cdots & 0 \\ L_{31} & L_{32} & 1 & 0 & \cdots \\ & & & & \end{bmatrix}$$

First equation is $c_1 = b_1$

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First equation is $c_1 = b_1$, no operations.

Second equation is $L_{21}c_1 + c_2 = b_2$

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Third equation is $L_{31}c_1 + L_{32}c_2 + c_3 = b_3$

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j^{th} equation: $2(j-1)$

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U

$$\sum_{k=1}^{n} k = \frac{k(k+1)}{2}$$

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j^{th} equation: $2(j-1)$ operations.

Grand total:

$$\sum_{j=1}^{n} 2(j-1) = 2\sum_{j=0}^{n-1} j = 2\frac{n(n-1)}{2} = n^2 - n$$

w $\quad$ w

$w = j-1$

Upper triangular solve

U upper triangular, nonzeros on diagonal.

$$U\mathbf{x} = \mathbf{c}$$

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Last equation is $U_{nn}x_n = c_n$, one operation.

$(n-1)^{\text{st}}$ equation is $U_{n-1,n-1}x_{n-1} + U_{n-1,n}x_n = c_{n-1}$

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$(n-2)^{\text{nd}}$ equation $U_{n-2,n-2}x_{n-2} + U_{n-2,n-1}x_{n-1} + U_{n-2,n}x_n = c_{n-2}$

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$(n-2)^{\text{nd}}$ equation $U_{n-2,n-2}x_{n-2} + U_{n-2,n-1}x_{n-1} + U_{n-2,n}x_n = c_{n-2}$: five operations

$(n-j)^{\text{th}}$ equation:

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$(n-j)^{\text{th}}$ equation:$2(j-1)+1$ operations.

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$(n-j)^{\text{th}}$ equation:$2(j-1)+1$ operations.

Grand total:

$$\sum_{j=1}^{n}[2(j-1)+1] == n^2 - n + n = n^2$$

L and U Solves are cheap

$Ax = b$

$A = LU$

$Lc = b$

$Ux = c$

One L solve and one U solve together: $\underbrace{2n^2 - n}$ operations.

same cost as $B \cdot b$

$n \times n$

$\underbrace{A^{-1}b}$

L and U Solves are cheap

One L solve and one U solve together: $2n^2 - n$ operations.

One matrix-vector multiplication: $2n^2 - 2$ operations!

LU factorization

$$A = \begin{bmatrix} a_{11} & * & \cdots & * \\ a_{21} & * & \cdots & * \\ \vdots & & & \\ a_{n1} & * & \cdots & * \end{bmatrix} \qquad -\frac{a_{21}}{a_{11}} = \ell$$

Clearing the first column:

For the second row, determine the multiple (one division)

Then subtract a multiple of row 1 (n-1 multiplications, n-1 subtractions)

$2(n-1)+1$

You do this for each of the n-1 rows below the first row.

$\hookrightarrow$ n-1 times

Clear first column:

$$(n-1)\left(2(n-1)+1\right) = 2(n-1)^2 + (n-1)$$

LU factorization

Clearing the first column:

For the second row, determine the multiple (one division)

Then subtract a multiple of row 1 (n-1 multiplications, n-1 subtractions)

You do this for each of the n-1 rows below the first row.

Total: $(n-1)(2(n-1)+1) = 2(n-1)^2 + (n-1)$

LU factorization

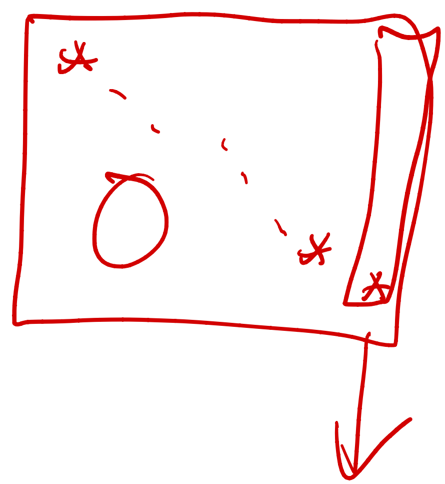
Clearing the first column: $2(n-1)^2 + (n-1)$

Clearing the second column is just like clearing the first column of a $(n-1) \times (n-1)$ matrix.

$2(n-2)^2 + (n-2)$ operations.

$\boxed{j^{th}: \quad 2(n-j)^2 + (n-j)}$

n'th



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Clearing the last column: $2(n-n) + (n-n) = 0$ operations!

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Grand total

$$\sum_{j=1}^{n} 2\underbrace{(j-1)}_{n-j}^2 + \underbrace{(j-1)}_{n-j} = \sum_{j=1}^{n-1} 2j^2 + j = 2\frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2} = \frac{2}{3}n^3 + a_2 n^2 + a_1$$

$j-1 = w-n+1$

$\sum_{j=1}^{n}(j-1)$ $\quad 0 + 1 + \cdots + n-1$

$\sum_{j=1}^{n}(n-j) = (n-1) + \cdots + 0$

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$$\frac{2}{3}n^3 + a_2 n^2 + a_1 n$$