Polynomial Interpolation

Math 426

University of Alaska Fairbanks

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Interpolation

We have data $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ that are samples of a function $y = f(x)$ and wish to estimate $f(x)$ for $x \neq x_0, \ldots, x_n$.

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One approach: find a polynomial $p(x)$ with $p(x_k) = y_k$, $k = 0, \ldots, n$.

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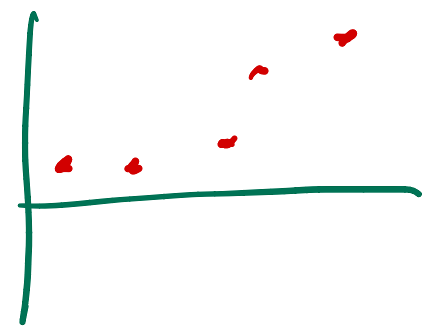

What order?

$$p(x) = c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0$$

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What order?

$$p(x) = c_m x^m + c_{m-1} x^{m-1} + \cdots + c_1 x + c_0$$

$m + 1$ coefficeints and $n + 1$ conditions $p(x_k) = y_k$, so n^{th} order is appropriate.

Polynomial Interpolation

$$p(x) = c_n x^n + \cdots + c_1 x + c_0$$

Polynomial Interpolation

$$p(x) = c_n x^n + \cdots + c_1 x + c_0$$

Equations to solve: $p(x_k) = y_k$

$$\begin{aligned}
c_0 + c_1 x_0 + \cdots + c_n x_0^n &= y_0 \\
c_0 + c_1 x_1 + \cdots + c_n x_1^n &= y_1 \\
\vdots \qquad\qquad &= \vdots \\
c_0 + c_1 x_n + \cdots + c_n x_n^n &= y_n
\end{aligned}$$

Polynomial Interpolation

$$p(x) = c_n x^n + \cdots + c_1 x + c_0$$

Equations to solve:

$$\begin{aligned} c_0 + c_1 x_0 + \cdots + c_n x_0^n &= y_0 \\ c_0 + c_1 x_1 + \cdots + c_n x_1^n &= y_1 \\ \vdots &= \vdots \\ c_0 + c_1 x_n + \cdots + c_n x_n^n &= y_n \end{aligned}$$

Matrix form

$(n+1) \times (n+1)$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Polynomial Interpolation

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Equations to solve:

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vander

vander(x)

Matrix form

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

(x.**n, ..., x, ones)

The matrix is a **Vandermonde** matrix.

Matlab Convention

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

vs.

$$\begin{pmatrix} x_0^n & \cdots & x_0^2 & x_0 & 1 \\ x_1^n & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & & \vdots & \\ x_n^n & \cdots & x_n^2 & x_n & 1 \end{pmatrix} \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Matlab Convention

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

vs.

$$\begin{pmatrix} x_0^n & \cdots & x_0^2 & x_0 & 1 \\ x_1^n & \cdots & x_1^2 & x_1 & 1 \\ \vdots & \ddots & & \vdots & \\ x_n^n & \cdots & x_n^2 & x_n & 1 \end{pmatrix} \begin{pmatrix} c_n \\ c_{n-1} \\ \vdots \\ c_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$5x^2 - 7x + 2$$

```
c = [5,-7,2]; x=[1,2,3]; polyval(c,x)
```

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix:

$$\left.\begin{matrix} x_0^3 \\ \vdots \\ x_n^3 \end{matrix}\right\}$$

n mult per col beyond col 2

Computational Effort

$$\begin{array}{c} \begin{matrix} & & (n+1) & & (n+1) \end{matrix} \\ \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \end{array} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

$(n+1)(n-1) = n^2 + O(n)$

Computational Effort

$(n+1) \times (n+1)$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

Solution of system:

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

Solution of system:$2/3n^3 + O(n^2)$

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

Solution of system: $2/3n^3 + O(n^2)$

$c_2 \cdot x \cdot x$

Evaluation:

$$p(x) = c_0 + c_1 x + \underline{c_2 x^2} + \cdots + c_n x^n$$

n additions

$0 + 1 + 2 \ldots \ldots + n$ $\quad \frac{n(n+1)}{2}$

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

Solution of system:$2/3n^3 + O(n^2)$

Evaluation:

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

$n-1$ additions, $0+1+2+\cdots+n = n(n+1)/2$ multiplications

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Construction of matrix: $n(n-1) = n^2 + O(n)$

Solution of system:$2/3n^3 + O(n^2)$

Evaluation:

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

$n-1$ additions, $0+1+2+\cdots+n = n(n+1)/2$ multiplications

$n^2/2 + O(n)$ operations.

Computational Effort

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ & \vdots & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

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$n-1$ additions, $0+1+2+\cdots+n = n(n+1)/2$ multiplications

$n^2/2 + O(n)$ operations.

Aim: Do better!

Lagrange Form

Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

$(x-x_1)(x-x_2)(x-x_3)\cdots(x-x_n)$

$$\phi_0(x) = \frac{(x - x_1)\cdots(x - x_n)}{(x_0 - x_1)\cdots(x_0 - x_n)}$$

↳ n^{th} order polynomial.

Lagrange Form

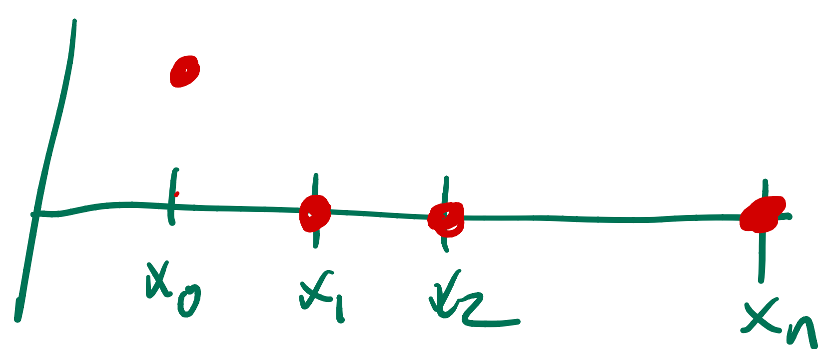
Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

$$\phi_0(x) = \frac{(x - x_1)\cdots(x - x_n)}{(x_0 - x_1)\cdots(x_0 - x_n)}$$

$$\phi_1(x) = \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)}$$

$\phi_0(x_1) = 0; \quad \phi_0(x_2) = 0; \ldots, \quad \phi_0(x_n) = 0$

$\phi_0(x_0) = 1$



Lagrange Form

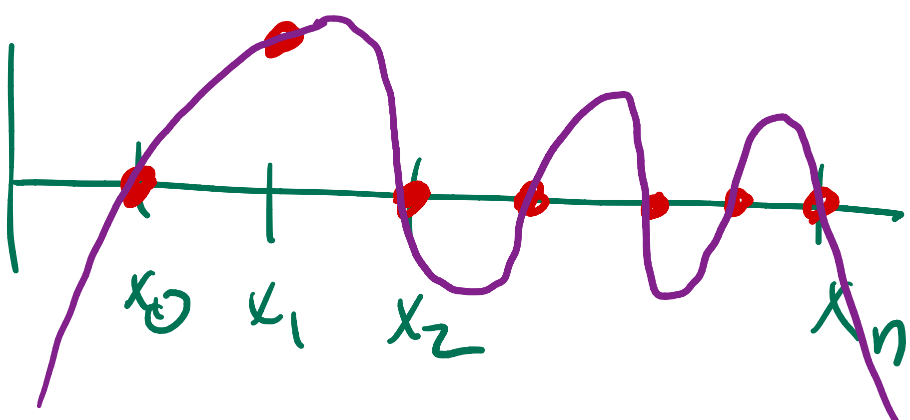
Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

Basis functions

$$\phi_0(x) = \frac{(x - x_1)\cdots(x - x_n)}{(x_0 - x_1)\cdots(x_0 - x_n)}$$

$$\phi_1(x) = \frac{(x - x_0)(x - x_2)\cdots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_0 - x_n)}$$

$\phi_1(x_0) = 0$; $\phi_1(x_1) = 1$; $\phi_1(x_2) = 0$ … , $\phi_1(x_n) = 0$



Lagrange Form

Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

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$$\phi_k(x) = \prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)}$$

Lagrange Form

Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

$$\phi_0(x) = \frac{(x - x_1)\cdots(x - x_n)}{(x_0 - x_1)\cdots(x_0 - x_n)}$$

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$$\phi_k(x) = \prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)}$$

$$\phi_k(x_j) = \begin{cases} 1 & \overset{k}{\cancel{i}} = j \\ 0 & \underset{k}{\cancel{i}} \neq j \end{cases}$$

Lagrange Form

Data points: $(x_0, y_0), \ldots, (x_n, y_n)$

$$\phi_k(x) = \prod_{j \neq k} \frac{(x - x_j)}{(x_k - x_j)}$$

$$p(x) = \sum_{k=0}^{n} y_k \phi_k(x)$$

$$p(x_0) = y_0 \underbrace{\phi_0(x_0)}_{1} + y_1 \underbrace{\phi_1(x_0)}_{0} + \cdots + y_n \underbrace{\phi_n(x_0)}_{0}$$

$$= y_0$$