Alternative Approach

Suppose we can find $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$ and such that

$$A = [\mathbf{q}_1 \ldots, \mathbf{q}_m] R$$

for an invertible upper triangular matrix R.

$$\begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

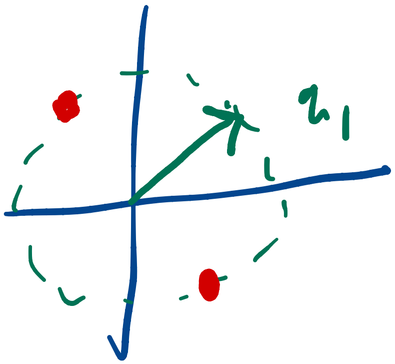
Alternative Approach

Suppose we can find $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$ and such that

$$A = [\mathbf{q}_1 \ldots, \mathbf{q}_m]R$$

for an invertible upper triangular matrix R.

QR factorization



$$Q = [\mathbf{q}_1 \ldots, \mathbf{q}_m]; \qquad Q^T Q = I$$

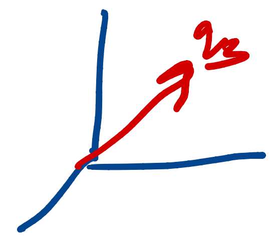
$\rightarrow A = QR$

$q_3 \cdot q_3 = 1$

$q_3 \cdot q_1 = 0$

$\hookrightarrow q_i \perp q_j$ if $i \neq j$

$q_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$



$q_3 \cdot q_3 = a^2 + b^2 + c^2$

$\vec{x} \cdot \vec{x} = \|\vec{x}\|_2^2 \leftarrow (\text{length})^2$ of $\vec{x}$

Alternative Approach

Suppose we can find $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$ and such that

$$A = [\mathbf{q}_1 \ldots, \mathbf{q}_m]R$$

for an invertible upper triangular matrix R.

$A\vec{x} = y$

$A^T A\vec{x} = A^T \vec{y}$

$$Q = [\mathbf{q}_1 \ldots, \mathbf{q}_m]; \qquad Q^T Q = I$$

$1 \quad 0 \text{———} 0$

$$Q^T \begin{bmatrix} q_1^T \\ \vdots \\ q_m^T \end{bmatrix} \begin{bmatrix} q_1 \cdots q_m \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \cdots & q_1^T q_m \\ q_2^T q_1 & \ddots & q_2^T q_m \\ \vdots & & \ddots \end{bmatrix}$$

$A = QR \qquad Q^T Q = I$

Alternative Approach

Suppose we can find $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$ and such that

$$A = [\mathbf{q}_1 \ldots, \mathbf{q}_m]R$$

for an invertible upper triangular matrix R.

$$Q = [\mathbf{q}_1 \ldots, \mathbf{q}_m]; \qquad Q^T Q = I$$

$$A = QR$$

$$A^T = (QR)^T = R^T Q^T$$

Normal equation:

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Becomes

$$(QR)^T (QR)\mathbf{x} = (QR)^T\mathbf{b}$$

$$R^T Q^T Q R \vec{x} = R^T Q^T b$$

$$R^T R \vec{x} = R^T Q^T \vec{b}$$

Alternative Approach

Suppose we can find $\mathbf{q}_1, \ldots, \mathbf{q}_m$ that satisfy $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$ and such that

$$A = [\mathbf{q}_1 \ldots, \mathbf{q}_m]R$$

for an invertible upper triangular matrix R.

Q $n \times n$

R $n \times n$

$$Q = [\mathbf{q}_1 \ldots, \mathbf{q}_m]; \qquad Q^T Q = I$$

Normal equation:

$$A^T A\mathbf{x} = A^t \mathbf{b}$$

Becomes

$$(QR)^T (QR)\mathbf{x} = (QR)^T \mathbf{b}$$

Then

$O(n^2)$ $O(n^2)$ $O(n^2)$

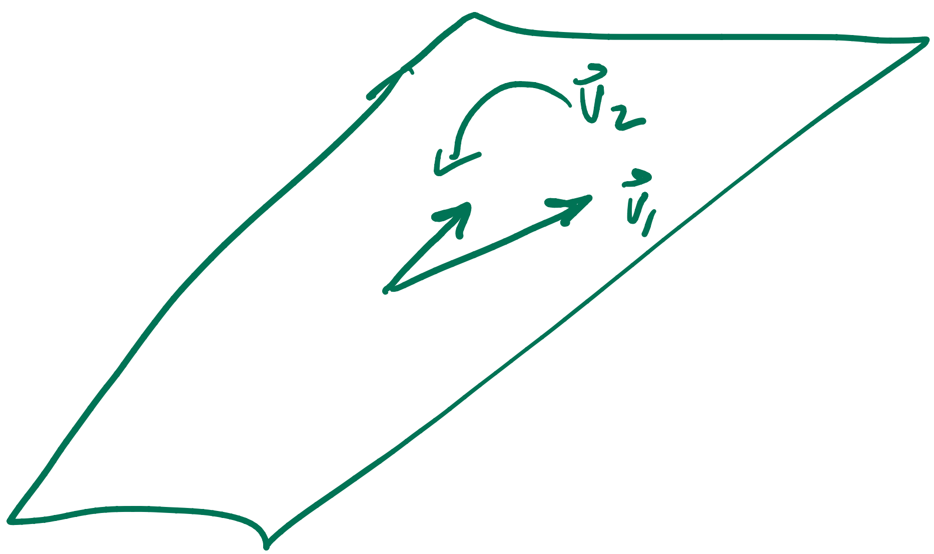
$$R\mathbf{x} = Q^T \mathbf{b}.$$

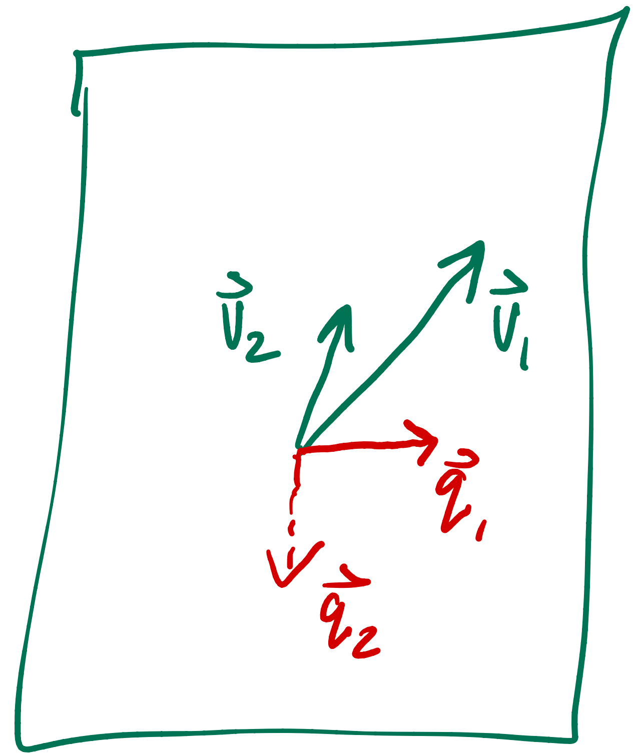
We avoid forming $A^T A$ or anything that looks like a square of A.

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.





Gramm-Schmidt

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$$\tilde{\mathbf{q}}_1 = \mathbf{v}_1$$

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$$\tilde{\mathbf{q}}_1 = \mathbf{v}_1$$

ensures $\vec{q}_1 \cdot \vec{q}_1 = 1$

$$\mathbf{q}_1 = \tilde{\mathbf{q}}_1/||\tilde{\mathbf{q}}_1||_2$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

$$\tilde{\mathbf{q}}_1 = \mathbf{v}_1$$

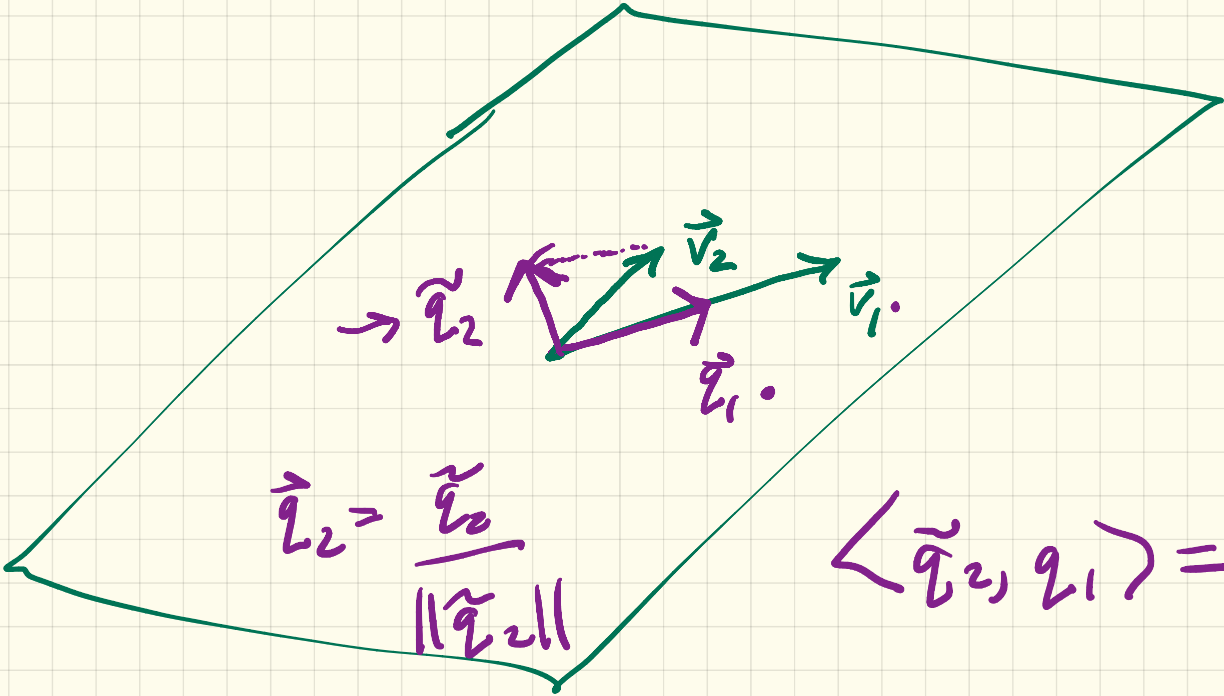
$$\vec{\tilde{q}}_1 = \vec{v}_1 \Big/ \|\tilde{q}_1\|_2$$

$$\mathbf{q}_1 = \tilde{\mathbf{q}}_1 / \|\tilde{\mathbf{q}}_1\|_2$$

$$\vec{v}_1 = \|\tilde{q}_1\|_2 \cdot \vec{q}_1$$

$$r_{11} = \|\tilde{\mathbf{q}}_1\|_2$$

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$



$$\vec{q}_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$

$$\langle \tilde{q}_2, q_1 \rangle = 0$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

$$\tilde{\mathbf{q}}_2 = \mathbf{v}_2 - \langle \mathbf{q}_1, \mathbf{v}_2 \rangle \mathbf{q}_1$$

$q_1^T v_2$

$\underline{q}_1 \cdot v_2$

$\langle q_1, v_2 \rangle$

$$\langle \hat{q}_2, q_1 \rangle = \langle v_2 - \langle q_1, v_2 \rangle q_1, q_1 \rangle$$

$$= \langle v_2, q_1 \rangle - \langle q_1, v_2 \rangle \underbrace{\langle q_1, q_1 \rangle}_{1}$$

$$= 0$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

$$\tilde{\mathbf{q}}_2 = \mathbf{v}_2 - \langle \mathbf{q}_1, \mathbf{v}_2 \rangle \, \mathbf{q}_1$$

$$\langle \mathbf{q}_2, \mathbf{q}_1 \rangle = \langle \mathbf{v}_2, \mathbf{q}_1 \rangle - \langle \mathbf{q}_1, \mathbf{v}2 \rangle \, ||\mathbf{q}_1||_2^2 = 0$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

$$\tilde{\mathbf{q}}_2 = \mathbf{v}_2 - \langle \mathbf{q}_1, \mathbf{v}_2 \rangle \, \mathbf{q}_1$$

$$\langle \mathbf{q}_2, \mathbf{q}_1 \rangle = \langle \mathbf{v}_2, \mathbf{q}_1 \rangle - \langle \mathbf{q}_1, \mathbf{v}2 \rangle \, ||\mathbf{q}_1||_2^2 = 0$$

$$\mathbf{q}_2 = \tilde{\mathbf{q}}_2 / ||\tilde{\mathbf{q}}_2||_2$$

$$\tilde{q}_2 = ||\tilde{q}_2||_2 \cdot q_2$$

$$\vec{v}_2 = \overbrace{\langle q_1, v_2 \rangle}^{r_{12}} q_1 + \overbrace{||\tilde{q}_2||_2}^{r_{22}} \cdot q_2$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

$$\tilde{\mathbf{q}}_2 = \mathbf{v}_2 - \langle \mathbf{q}_1, \mathbf{v}_2 \rangle \mathbf{q}_1$$

$$\langle \mathbf{q}_2, \mathbf{q}_1 \rangle = \langle \mathbf{v}_2, \mathbf{q}_1 \rangle - \langle \mathbf{q}_1, \mathbf{v}2 \rangle ||\mathbf{q}_1||_2^2 = 0$$

$$\mathbf{q}_2 = \tilde{\mathbf{q}}_2 / ||\tilde{\mathbf{q}}_2||_2$$

$$\mathbf{v}_2 = ||\tilde{\mathbf{q}}_2||_2 \mathbf{q}_2 + \langle \mathbf{q}_1, \mathbf{v}2 \rangle \mathbf{q}_1$$

Gramm-Schmidt

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$, linearly independent.

Goal: Find perpendicular unit vectors $\mathbf{q}_1$, $\mathbf{q}_2$ such that the span of the $\mathbf{v}$'s is the same as the span of the $\mathbf{q}$'s.

Already found $\mathbf{q}_1$, r_{11}

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$

$$\tilde{\mathbf{q}}_2 = \mathbf{v}_2 - \langle \mathbf{q}_1, \mathbf{v}_2 \rangle \mathbf{q}_1$$

$$\langle \mathbf{q}_2, \mathbf{q}_1 \rangle = \langle \mathbf{v}_2, \mathbf{q}_1 \rangle - \langle \mathbf{q}_1, \mathbf{v}2 \rangle ||\mathbf{q}_1||_2^2 = 0$$

$$\mathbf{q}_2 = \tilde{\mathbf{q}}_2 / ||\tilde{\mathbf{q}}_2||_2$$

$$\mathbf{v}_2 = ||\tilde{\mathbf{q}}_2||_2 \mathbf{q}_2 + \langle \mathbf{q}_1, \mathbf{v}2 \rangle \mathbf{q}_1$$

$$r_{12} = \langle \mathbf{q}_1, \mathbf{v}2 \rangle ; \qquad r_{22} = ||\tilde{\mathbf{q}}_2||_2$$

$$[\vec{v}_1, \vec{v}_2]$$

v_1, v_2

q_1, q_2

$$v_1 = r_{11} q_1$$

$$v_2 = r_{12} q_1 + r_{22} q_2$$

$$[\vec{q}_1, \vec{q}_2] \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

$$= [r_{11}\vec{q}_1 + 0\vec{q}_2, \; r_{12}\vec{q}_1 + r_{22}\vec{q}_2]$$

Gramm-Schmidt: More vectors!

Start with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Compute $\mathbf{q}_1$, $\mathbf{q}_2$, r_{11}, r_{12} and r_{22} the same way.

Gramm-Schmidt: More vectors!

Start with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Compute $\mathbf{q}_1$, $\mathbf{q}_2$, r_{11}, r_{12} and r_{22} the same way.

$$\tilde{\mathbf{q}}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2$$

Want $\langle \tilde{q}_3, q_1 \rangle = 0$, $\langle \tilde{q}_3, q_2 \rangle = 0$.

$$\langle \tilde{q}_3, q_1 \rangle = \langle v_3, q_1 \rangle - \langle v_3, q_1 \rangle$$

$$= 0$$

Gramm-Schmidt: More vectors!

Start with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Compute $\mathbf{q}_1$, $\mathbf{q}_2$, r_{11}, r_{12} and r_{22} the same way.

$$\tilde{\mathbf{q}}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2$$

Verify that $\tilde{\mathbf{q}}_3$ is perpendicular to $\mathbf{q}_1$ and $\mathbf{q}_2$.

Gramm-Schmidt: More vectors!

Start with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Compute $\mathbf{q}_1$, $\mathbf{q}_2$, r_{11}, r_{12} and r_{22} the same way.

$$\tilde{\mathbf{q}}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2$$

Verify that $\tilde{\mathbf{q}}_3$ is perpendicular to $\mathbf{q}_1$ and $\mathbf{q}_2$.

Make it have unit length:

$$\mathbf{q}_3 = \tilde{\mathbf{q}}_3 / ||\tilde{\mathbf{q}}_3||_2$$

$$\mathbf{v}_3 = \langle \mathbf{v}_3, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{v}_3, \mathbf{q}_2 \rangle \mathbf{q}_2 + ||\tilde{\mathbf{q}}_3||_2 \mathbf{q}_3$$

$$r_{13} = \langle \mathbf{q}_1, \mathbf{v}_3 \rangle ; \qquad r_{23} = \langle \mathbf{q}_2, \mathbf{v}_3 \rangle ; \qquad r_{33} = ||\tilde{\mathbf{q}}_3||_2$$

Gramm Schmidt = QR Factorization

Starting from $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$:

$$\begin{aligned}
\mathbf{v}_1 &= r_{11}\mathbf{q}_1 \\
\mathbf{v}_2 &= r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2 \\
\mathbf{v}_3 &= r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 + r_3 3\mathbf{q}_3
\end{aligned}$$

Gramm Schmidt = QR Factorization

Starting from $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$:

$$\begin{aligned} \mathbf{v}_1 &= r_{11}\mathbf{q}_1 \\ \mathbf{v}_2 &= r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2 \\ \mathbf{v}_3 &= r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 + r_33\mathbf{q}_3 \end{aligned}$$

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \underbrace{[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]}_{18 \times 3} \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

$A = [\vec{v}_1, .., \vec{v}_3]$ 18×3

Gramm Schmidt = QR Factorization

Starting from $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \\ 12 \end{bmatrix}$$

A $\qquad$ b

$$\mathbf{v}_1 = r_{11}\mathbf{q}_1$$
$$\mathbf{v}_2 = r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2$$
$$\mathbf{v}_3 = r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 + r_33\mathbf{q}_3$$

$\|Ax-b\|_2$

$\rightarrow A = QR$

$Rx = Q^Tb$

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix}$$

If $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$,

$$A = QR$$
$$Q^TQ = I$$

$A\backslash b \rightarrow x$

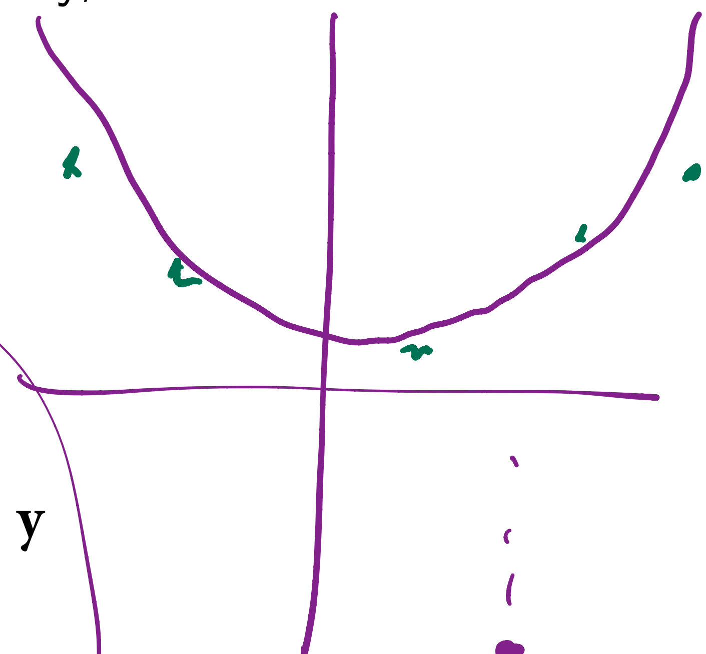
Quadratic Interpolation

Data points: $(x_1, y_1), \ldots, (x_5, y_5)$.

Let's find a 'best fit' quadratic polynomial. Ideally, find $\mathbf{c} = [c_0, c_1, c_2]^T$ such that

$$c_0 + c_1 x_k + c_2 x_k^2 = y_k$$

for each k.



$$V\mathbf{c} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{pmatrix} \mathbf{c} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \mathbf{y}$$

We'll minimize

$$||V\mathbf{c} - \mathbf{y}||_2^2$$

instead.