Condition Number

$$\kappa = ||A||_p ||A^{-1}||_p$$

is the called the condition number of A (with respect to the p norm).

When solving $A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$,

$$\frac{||\Delta\mathbf{x}||_p}{||x||_p} \leq \kappa \frac{||\Delta\mathbf{b}||_p}{||b||_p}$$

It tells you how relative error scales from the input $\mathbf{b}$ to the output $\mathbf{x}$.

Matlab Demo

$$\|A\|_1 = \max \|Aw\|_1 \text{ where } \|w\|_1 = 1$$

$$p \qquad\qquad p \qquad\qquad p$$

$$\|A\|_2$$

Error in A

Suppose we solve:

$$(A + \Delta A)(\mathbf{x} + \Delta \mathbf{x}) = \mathbf{b}$$

The relative error in $\mathbf{x}$ is still controlled by the condition number:

$$\frac{||\Delta x||}{||x||} \leq \kappa \cancel{\frac{1}{1 - \kappa ||\Delta A|| / ||A||}} \frac{||\Delta A||}{||A||}$$

Conditioning vs. Stability

Recall

$$A = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix} \qquad A^{-1} = -\frac{1}{1 - 10^{-10}} \begin{pmatrix} 1 & -1 \\ -1 & 10^{-20} \end{pmatrix}$$

$||A||_\infty = 2$ and $||A^{-1}||_\infty \approx 2$ so $\kappa(A) \approx 4$. This matrix is well conditioned. We expect when solving $A\mathbf{x} = \mathbf{b}$ that

$$\frac{||\Delta\mathbf{x}||_\infty}{||x||_\infty} \approx \kappa\epsilon$$

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1 vs 10^{-16}

Instead: found a relative error of size 1.

Conditioning vs. Stability

12-2

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Instead: found a relative error of size 1.

We did a lousy job even though we could have been reasonably expected to do a good job. This is an example of **instability**.

Backward Stability

Loosely, an algorithm for solving

$$A\mathbf{x} = \mathbf{b}$$

is **backwards stable** if it generates a solution $\hat{\mathbf{x}}$ with

$$A\hat{\mathbf{x}} = \hat{\mathbf{b}}$$

with $\hat{\mathbf{b}}$ "near" $\mathbf{b}$. For a backwards stable algorithm

$$\frac{||\overset{\hat{x}}{\cancel{x}} - \mathbf{x}|||}{||\mathbf{x}||} \leq C\kappa(A)$$

An algorithm violates this if we can find A_n, $\kappa(A_n)$ stays bounded, but relative errors blow up. This is instablilty.

Mixed news

- Gaussian elimination with partial pivoting is **not** backwards stable for every square matrix.
- The exceptions are rare. Pick a random matrix, and we don't seem to land on them.
- "In 50 year of computiong, mo matrix problime sthat excite an explosive instability are known to have arisen under natural circumstances"

We treat Gaussian elimination with partial pivoting as stable even if it isn't.