Condition Numbers and Stability

Math 426

University of Alaska Fairbanks

October 19, 2020

HW 6: LU

$L^T L$

Bold answers

Code: please add tests:

$\boxed{\text{myinv}(A) \to B}$

inv(A)

$A \cdot B = I$

$A \cdot B - I \to \text{norm}(\cdot, \text{Inf})$

$\sim 10^{-16}$

$A \qquad B = [\vec{b}_1, \ldots, \vec{b}_n]$

$$AB = [A\vec{b}_1, \ldots, A\vec{b}_n] = [\vec{e}_1, \ldots, \vec{e}_n]$$

$$A\vec{b}_1 = \vec{e}_1$$

$$\vdots$$

$$Ab_n = \vec{e}_n$$

1) $LU = A$

2) Loop: on k

Solve $A\vec{b}_k = \vec{e}_k$

$$L\vec{c}_k = \vec{e}_k$$

$$Ub_k = \vec{c}_c$$

$O(n^3)$

1) $LU = A$

2) Loop: on k

Solve $A b_k = \vec{e}_k$

$$L\vec{c}_k = \vec{e}_k$$

$$U b_k = \vec{c}_k$$

$n = 10$

$n = 100$

$O(n^3)$

1) Loop:

a) $LU = A$

b) Solve

$$L\vec{c}_k = \vec{e}_k$$

$$U b_k = \vec{c}_k$$

$O(n^4)$

plsolve (P, L, b)

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_3 \\ b_1 \\ b_2 \end{pmatrix}$$

$$\vec{p} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

10000×10000

$P \to$ 300 Mb

$$\hat{b} = Pb$$

$$Lx = \hat{b}$$

Eigenvalue Refresher

A vector $\mathbf{x}$ is an **eigenvector** of A if there is a number λ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Picture:

$n \times n$ at most

n eigenvalues

Eigenvalue Refresher

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Picture:

e.g.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

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Picture:

e.g.

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e.g.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Computing the 2-norm

The 2-norm of a matrix is the square root of the largest eigenvalue of $A^T A$.

$$||A\mathbf{w}||_2^2 = (A\mathbf{w}) \cdot (A\mathbf{w}) = \mathbf{w}^T A^T A\mathbf{w}$$

If $A^T A\mathbf{w} = \lambda\mathbf{w}$ then

$$\mathbf{w}^T A^T A\mathbf{w} = \lambda\mathbf{w}^T\mathbf{w} = \lambda||\mathbf{w}||_2^2$$

So

$$\frac{||A\mathbf{w}||_2}{||\mathbf{w}||_2} = \sqrt{\lambda}$$

$||A||_1 \Rightarrow$ max 1-norm of cols

$||A||_\infty \Rightarrow$ max 1-norm of rows

$A = [$

Computing the 2-norm

$\lambda = 22.9$

$\lambda = 68.1$

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{pmatrix}$$

$\|A\|_2 \approx 8.2$

$$A^T A = \cdots$$

$$A^T A = \begin{bmatrix} 35 & 20 \\ 20 & 56 \end{bmatrix}$$

$$A^T A x = \lambda x$$

$$(A^T A - \lambda I) x = 0$$

$$\left| \begin{bmatrix} 35-\lambda & 20 \\ 20 & 56-\lambda \end{bmatrix} \right| = (35-\lambda)(56-\lambda) - 400$$

$$= 0$$

Computing the 2-norm

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$A^T A = \cdots$$

Now compute eigenvalues.

Condition Number

Want to solve $A\mathbf{x} = \mathbf{b}$.

Solve $A\mathbf{y} = \mathbf{b} + \Delta\mathbf{b}$ with $\mathbf{y} = \mathbf{x} + \Delta\mathbf{x}$

$$||\Delta\mathbf{x}|| = ||A^{-1}\Delta\mathbf{b}||$$
$$||\mathbf{b}|| = ||A\mathbf{x}||$$

For all $\mathbf{y}$ and $\mathbf{w}$:

$$||A^{-1}\mathbf{y}|| \leq ||A^{-1}||||\mathbf{y}||$$
$$||A\mathbf{w}|| \leq ||A||||\mathbf{w}||$$

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$$A\Delta x = \Delta b$$

$$\Delta x = A^{-1}\Delta b$$

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$$||\mathbf{b}|| = ||A\mathbf{x}||$$

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$$||Ax|| \leq ||A|| \, ||x||$$

$$\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \leq ||A^{-1}||\frac{||\Delta\mathbf{b}||}{||\mathbf{x}||} \leq ||A^{-1}||\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}\frac{||\mathbf{b}||}{||\mathbf{x}||} = ||A^{-1}||\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}\frac{||A\mathbf{x}||}{||\mathbf{x}||} \leq ||A||||A^{-1}||\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$$

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Condition Number

p, A $\quad\quad \kappa_p(A)$

$$\kappa = ||A||_p||A^{-1}||_p$$

is the called the condition number of A (with respect to the p norm).

When solving $A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$,

ε

$$\frac{||\Delta\mathbf{x}||_p}{||x||_p} \leq \kappa \frac{||\Delta\mathbf{b}||_p}{||b||_p}$$

It tells you how relative error scales from the input $\mathbf{b}$ to the output $\mathbf{x}$.

Smaller is better!

Matlab Demo

Error in A

Suppose we solve:

$$(A + \Delta A)(\mathbf{x} + \Delta \mathbf{x}) = \mathbf{b}$$

The relative error in $\mathbf{x}$ is still controlled by the condition number:

$$\frac{||\Delta x||}{||x||} \leq \kappa \cancel{\frac{1}{1 - \kappa||\Delta A||/||A||}} \frac{||\Delta A||}{||A||}$$

Conditioning vs. Stability

Recall

$$A = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix} \qquad A^{-1} = -\frac{1}{1 - 10^{-10}} \begin{pmatrix} 1 & -1 \\ -1 & 10^{-20} \end{pmatrix}$$

$||A||_\infty = 2$ and $||A^{-1}||_\infty \approx 2$ so $\kappa(A) \approx 4$. This matrix is well conditioned. We expect when solving $A\mathbf{x} = \mathbf{b}$ that

$$\frac{||\Delta\mathbf{x}||_\infty}{||x||_\infty} \approx \kappa\epsilon$$

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1 vs 10^{-16}

Instead: found a relative error of size 1.

Conditioning vs. Stability

12-2

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Instead: found a relative error of size 1.

We did a lousy job even though we could have been reasonably expected to do a good job. This is an example of **instability**.