Condition Numbers and Stability

Math 426

University of Alaska Fairbanks

October 16, 2020

How big is a vector?

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$$Ax = b$$

$$\uparrow$$

$$b + \Delta b$$

$$\mathbf{b} = \begin{pmatrix} 2 \\ -1/3 \end{pmatrix}$$

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$$\mathbf{b} = \begin{pmatrix} 2 \\ -1/3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

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2-norm:

$$||\mathbf{b}||_2 = \sqrt{b_1^2 + b_2^2} = \left(4 + \frac{1}{9}\right)^{\frac{1}{2}} \approx 2.03$$

Motivating Condition Number

$$||\Delta \mathbf{x}|| = ||A^{-1}\Delta \mathbf{b}||$$
$$||\mathbf{b}|| = ||A\mathbf{x}||$$

Suppose

$$||A^{-1}\mathbf{y}|| \leq M||\mathbf{y}||$$
$$||A\mathbf{w}|| \leq C||\mathbf{w}||$$

no matter what $\mathbf{y}$ and $\mathbf{w}$ are.

$Ax = b$

$Ay = b + \Delta b$

$\hookrightarrow y = x + \Delta x$

$A\Delta x = \Delta b$

$\Delta x = A^{-1}\Delta b$

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$C||x||$

$$\frac{||\Delta \mathbf{x}||}{||\mathbf{x}||} \leq M\frac{||\Delta \mathbf{b}||}{||\mathbf{x}||} \overset{=}{\cancel{\leq}} M\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}\frac{||\mathbf{b}||}{||\mathbf{x}||} = M\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}\frac{||A\mathbf{x}||}{||\mathbf{x}||} \leq CM\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}$$

Matrix Norms

Suppose

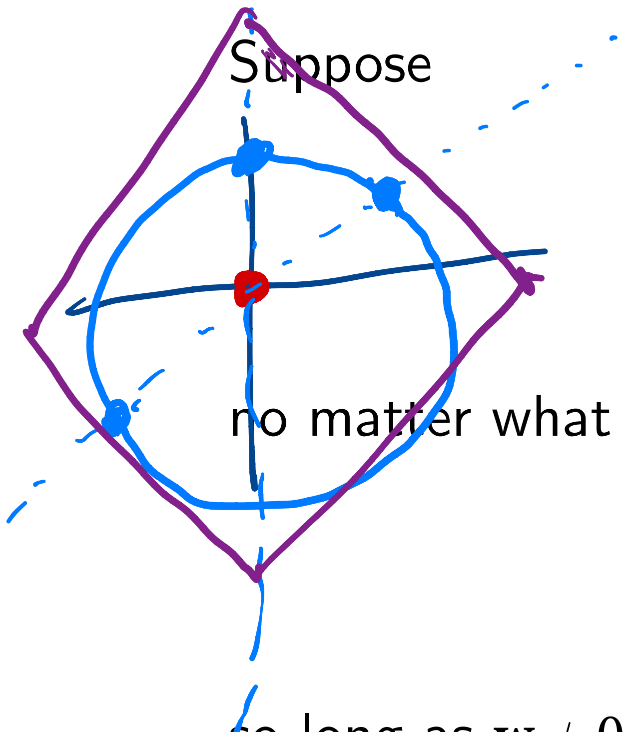
$$||A\mathbf{w}||_1 \leq C||\mathbf{w}||_1$$

no matter what $\mathbf{w}$ is. Then, if $\mathbf{w} \neq 0$,

$$\frac{||A\mathbf{w}||_1}{||\mathbf{w}||_1} \leq C$$

so long as $\mathbf{w} \neq 0$.

Matrix Norms



Suppose

$$||A\mathbf{w}||_1 \le C||\mathbf{w}||_1$$

$$\frac{||Acw||}{||cw||} = \frac{|c|\,||Aw||}{|c|\,||w||}$$

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$$\frac{||A\mathbf{w}||_1}{||\mathbf{w}||_1} \le C$$

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The 1-norm of A is the smallest C that works in this inequality.

$$||A||_1 = \max_{\mathbf{w}\neq 0}\left(\frac{||A\mathbf{w}||_1}{||\mathbf{w}||_1}\right).$$

$||w||_1 = 1$

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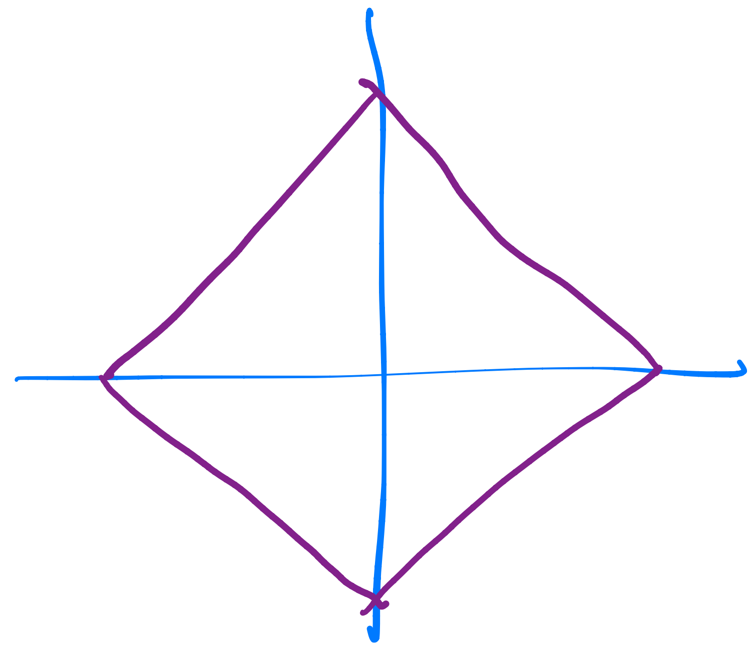

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The 1-norm of A is the smallest C that works in this inequality.

$$||A||_1 = \max_{\mathbf{w}\neq 0}\left(\frac{||A\mathbf{w}||_1}{||\mathbf{w}||_1}\right). \rightarrow \max_{||w||_1=1} ||Aw||_1$$

We only need to work with $||\mathbf{w}||_1 = 1$:

Matrix Norms

$$||A|| = \max_{||\mathbf{w}||=1} ||Aw||$$

What does this measure?

If you start with something (w) of length 1, how long can Aw possibly be?

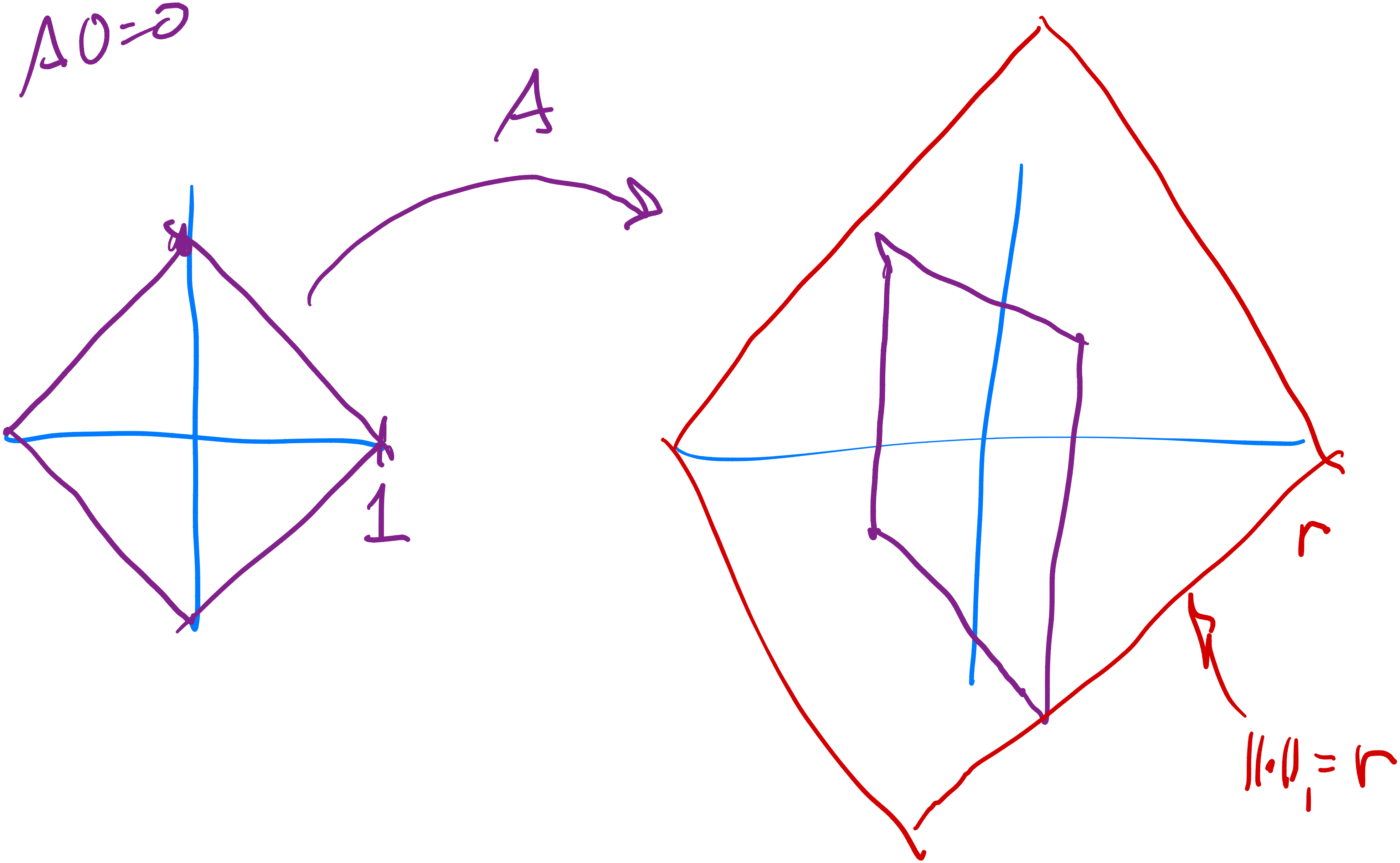
Matrix Norms

$$||A|| = \max_{||\mathbf{w}||=1} ||Aw||$$

What does this measure?

If you start with a size 1 vector, what's the largest length that A can make it grow (or shrink) to?

Picture: $\|A\|_1$



Picture: $\|A\|_\infty$

How to compute $||A||_1$

Suppose

$$A = [\mathbf{v}_1, \dots, \mathbf{v}_n]$$

and

$$||\mathbf{w}||_1 = 1$$

$\|A w\|_1$

$\|w\|_1 = 1$

$$A\vec{w} = [\vec{v}_1, \dots, \vec{v}_n] \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = w_1\vec{v}_1 + w_2\vec{v}_2 + \dots + w_n\vec{v}_n$$

How to compute $||A||_1$

Suppose

$$A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

$$||\mathbf{w}||_1 = 1$$

$||a+b||_1 \leq ||a||_1 + ||b||_1$

Let $M = \max(||\mathbf{v}_k||_1)$. Then

$$\begin{aligned} ||A\mathbf{w}||_1 &= ||w_1\mathbf{v}_1 + \cdots + w_n\mathbf{v}_n||_1 \\ &\leq |w_1|||\mathbf{v}_1||_1 + \cdots |w_n|||\mathbf{v}_n||_1 \\ &\leq |w_1|M + \cdots + |w_n|M \\ &= M \end{aligned}$$

$(|w_1| + \cdots + |w_n|)M$

$||w||_1 = |w_1| + \cdots + |w_n|$

$||w||_1 = 1$

$||a+b+c|| \leq ||a+b|| + ||c|| \leq ||a|| + ||b|| + ||c||$

How to compute $||A||_1$

Suppose

$$A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

and

$$||\mathbf{w}||_1 = 1$$

$Aw = v_k$

Let $M = \max(||\mathbf{v}_k||_1)$. Then

$$\begin{aligned} ||A\mathbf{w}|| &= ||w_1\mathbf{v}_1 + \cdots w_n\mathbf{v}_n|| \\ &\leq |w_1|||\mathbf{v}_1||_1 + \cdots |w_n|||\mathbf{v}_n|| \\ &\leq |w_1|M + \cdots + |w_n|M \\ &= M \end{aligned}$$

$$w = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow k^{th}$$

And if $M = ||\mathbf{v}_k||_1$ for some k, let $w = \mathbf{e}_k$ to get equality.

How to compute $||A||_1$

Suppose

$$A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$$

and

$$||\mathbf{w}||_1 = 1$$

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$$

Let $M = \max(||\mathbf{v}_k||_1)$. Then

$$\begin{aligned} ||A\mathbf{w}|| &= ||w_1\mathbf{v}_1 + \cdots w_n\mathbf{v}_n|| \\ &\leq |w_1|||\mathbf{v}_1||_1 + \cdots |w_n|||\mathbf{v}_n|| \\ &\leq |w_1|M + \cdots + |w_n|M \\ &= M \end{aligned}$$

9, 12

$||A||_1 = 12$

And if $M = ||\mathbf{v}_k||_1$ for some k, let $w = \mathbf{e}_k$ to get equality.

$$||A||_1 = \max_k(||\mathbf{v}_k||_1)$$

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\|w\|_1 = |0| + |1| = 1$$

$$Aw = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\|\cdot\|_1 = 12$$

How to compute $||A||_\infty$

$$\vec{w} = \begin{pmatrix} -3.2 \\ 6.1 \\ -7.8 \end{pmatrix}$$

Suppose

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\|\vec{w}\|_\infty = 7.8$$

and $\mathbf{w} = [w_1, w_2, \cancel{w_3}]^T$ has $||\mathbf{w}||_\infty = 1$.

$$\|Aw\|_\infty = \left\| \begin{bmatrix} 1 \cdot w_1 + 2w_2 \\ -3w_1 + 4w_2 \\ 5w_1 + 6w_2 \end{bmatrix} \right\|_\infty \begin{matrix} \le 3 \\ \le 7 \\ \le 11 \end{matrix}$$

$$3 = |1| + |2|$$

$$7 = |-3| + |4|$$

$$11 = |5| + |6|$$

$\|A\|_\infty \to$ max 1-norm of <u>rows</u> of A.

How to compute $||A||_\infty$

Suppose

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{pmatrix}$$

and $\mathbf{w} = [w_1, w_2, w_3]^T$ has $||\mathbf{w}||_\infty = 1$.

Let's compute $||A\mathbf{w}||_\infty$:

How to compute $||A||_\infty$

Suppose

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 5 & 6 \end{pmatrix}$$

and $\mathbf{w} = [w_1, w_2, w_3]^T$ has $||\mathbf{w}||_\infty = 1$.

Let's compute $||A\mathbf{w}||_\infty$:

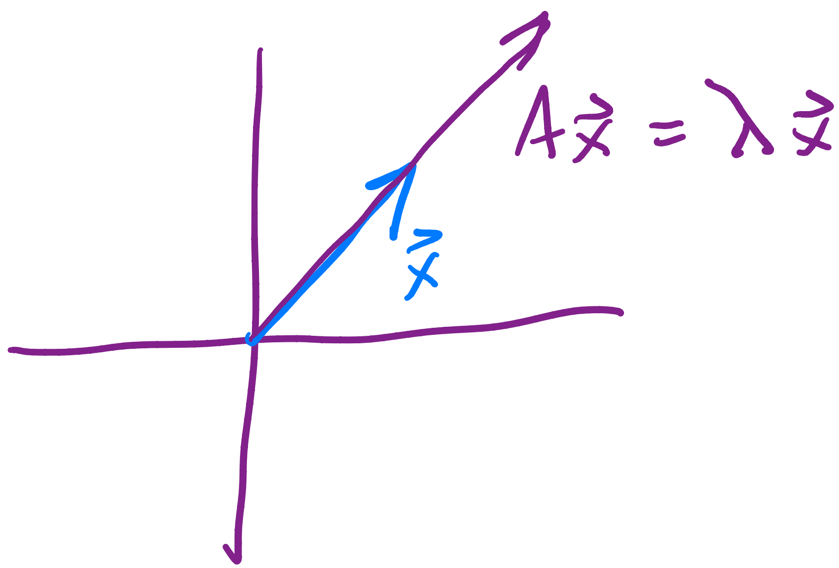
$||A||_\infty$ is the maximum 1-norm of the **rows** of A.

Eigenvalue Refresher

A vector $\mathbf{x}$ is an **eigenvector** of A if there is a number λ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Picture:



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eigenvalue

Picture:

λ

e.g.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad A\vec{x} = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \lambda = 1/2 \quad \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Eigenvalue Refresher

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Picture:

e.g.

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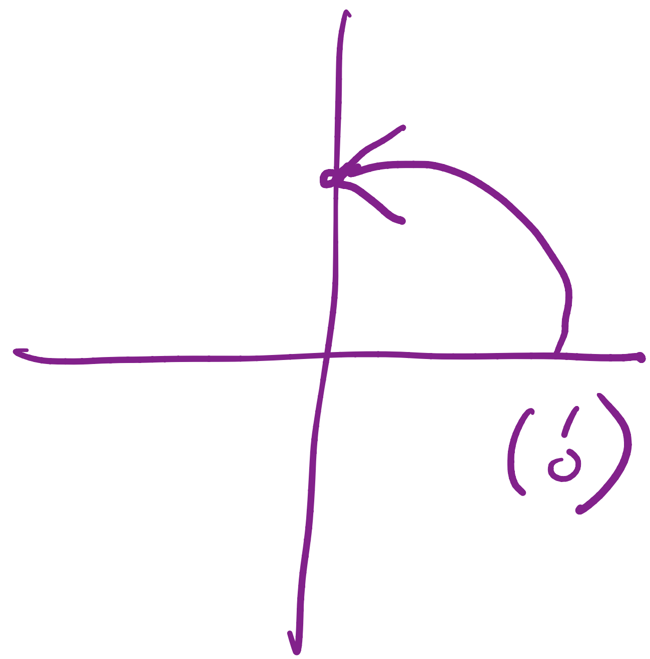
e.g.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$\begin{pmatrix} i \\ 1 \end{pmatrix}$

Computing the 2-norm

The 2-norm of a matrix is the square root of the largest eigenvalue of $A^T A$.

$(Aw)^T = w^T A^T$

$$\underbrace{||A\mathbf{w}||_2^2}_{} = (A\mathbf{w}) \cdot (A\mathbf{w}) = \underbrace{\mathbf{w}^T A^T A\mathbf{w}}_{}$$

If $A^T A\mathbf{w} = \lambda\mathbf{w}$ then

$||Aw||_2^2 = \lambda ||w||_2^2$

$$\mathbf{w}^T A^T A\mathbf{w} = \lambda\mathbf{w}^T\mathbf{w} = \lambda||\mathbf{w}||_2^2$$

$w \cdot w = ||w||_2^2$

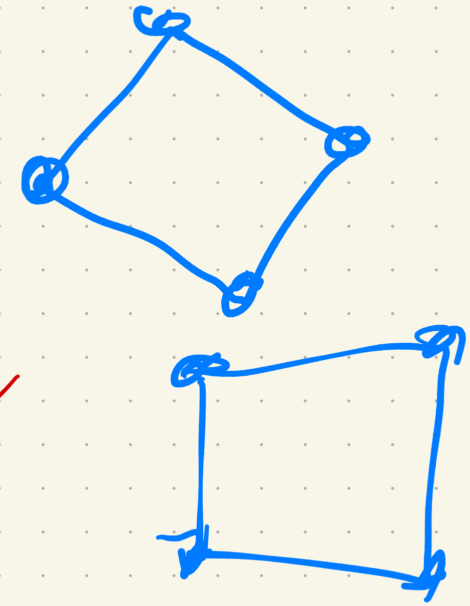
$||Aw||_2 = \sqrt{\lambda}\, ||w||_2$

So

$$\frac{||A\mathbf{w}||_2}{||\mathbf{w}||_w} = \sqrt{\lambda}$$

$||A||_2 \leq \sqrt{\lambda}$

$\|w\|_2 = 1$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



$B^T = B$

$(A^TA)^T = A^T \cdot (A^T)^T = A^TA$

$\rightarrow n \times n$ There is a basis of n orthorgonal eigen vectors.