Condition Numbers and Stability

Math 426

University of Alaska Fairbanks

October 14, 2020

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We discussed partial pivoting because we saw that roundoff error in A led to spectacular failures otherwise. But we have no proof that partial pivoting is a perfect remedy. (Spoiler alert: there is no perfect remedy)

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How big is the error $\Delta\mathbf{x}$ compared to the error $\Delta\mathbf{b}$?

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$$C = \frac{|\Delta x|}{|\Delta b|}$$

If $\Delta\mathbf{x}$ and $\Delta\mathbf{b}$ were numbers

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The value C would tell us how the error in $\mathbf{b}$ magnified (or shrunk!) to become the error $\Delta\mathbf{x}$. The constant C plays a role that would be called the **absolute condition number**.

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An ongoing theme: relative errors are more relevant than absolute errors.

$$\frac{|\Delta\mathbf{x}|}{|\mathbf{x}|} = \kappa\frac{|\Delta\mathbf{b}|}{|\mathbf{b}|}$$

Key concept: **relative condition number** (to be defined).

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$$\mathbf{b} = \begin{pmatrix} 2 \\ -1/3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \sqrt{b_1^2 + b_2^2}$$

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2-norm:

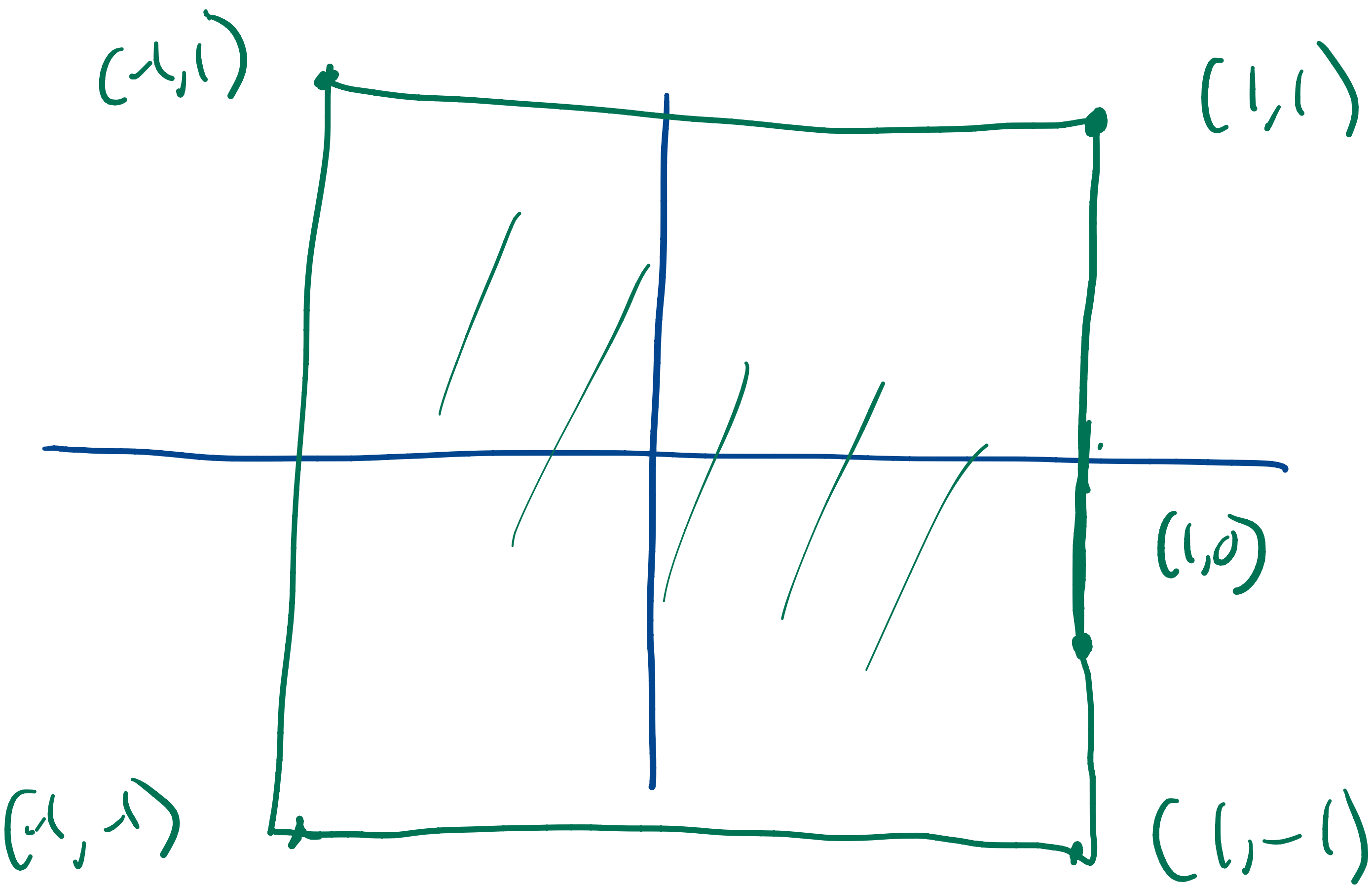
$$||\mathbf{b}||_2 = \sqrt{b_1^2 + b_2^2} = \left(4 + \frac{1}{9}\right)^{\frac{1}{2}} \approx 2.03$$

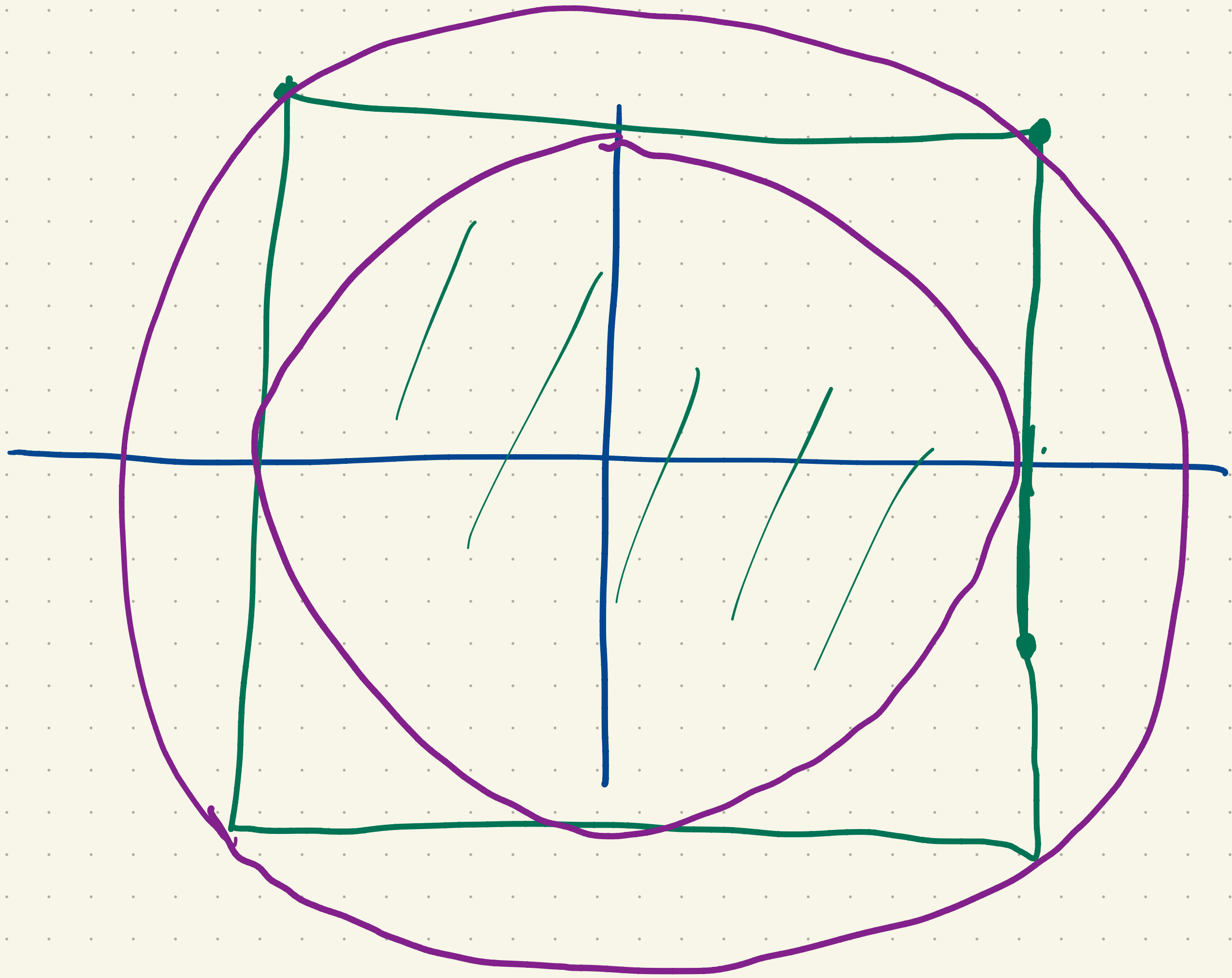
Unit Ball for the ∞-norm

$\mathbf{x} = (x_1, x_2)^T$

$$||\mathbf{x}||_\infty = \max(x_1, x_2)$$

Unit ball: $||\mathbf{x}||_\infty \leq 1$





Unit Ball for the 2-norm

$\mathbf{x} = (x_1, x_2)^T$

$\|x\|_2 \leq 1/2$

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2}$$

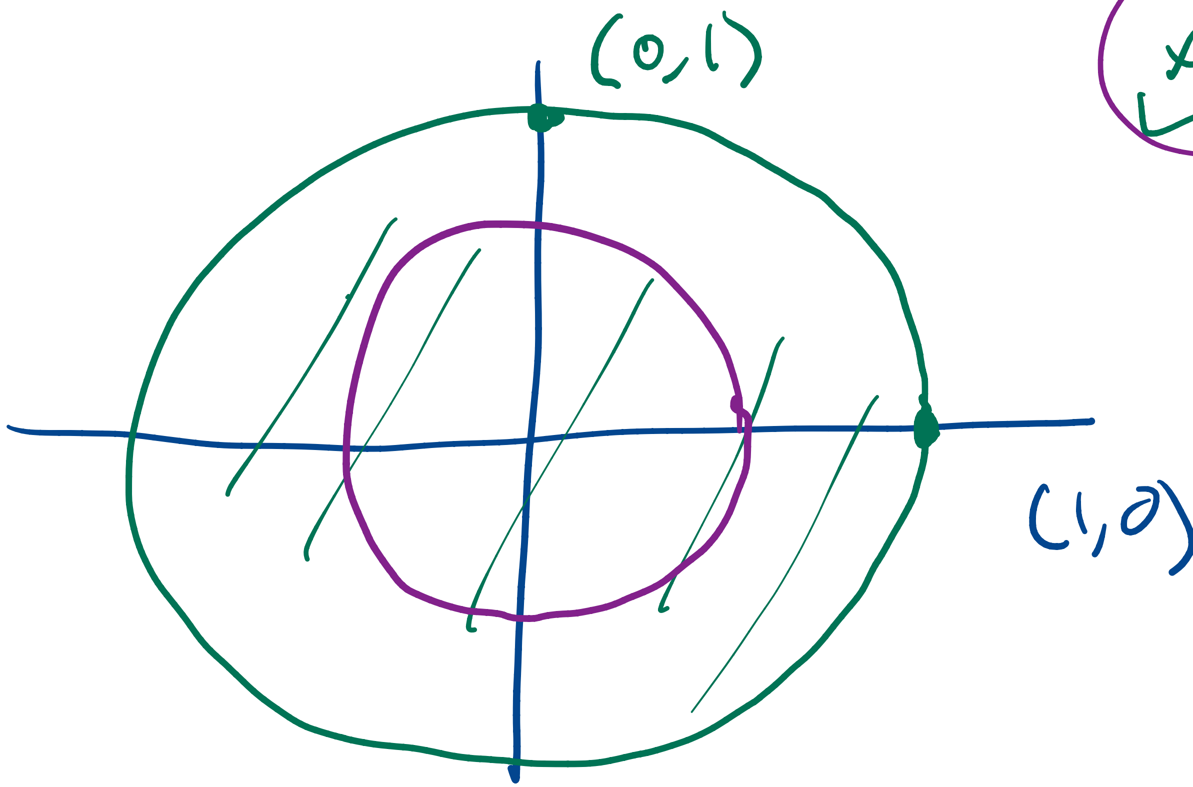
$\|x\|_2 = 1$

$\sqrt{x_1^2 + x_2^2} = 1$

$x_1^2 + x_2^2 = 1^2 = 1$

$x^2 + y^2 = 1$

Unit ball: $\|\mathbf{x}\|_2 \leq 1$



Unit Ball for the 1-norm

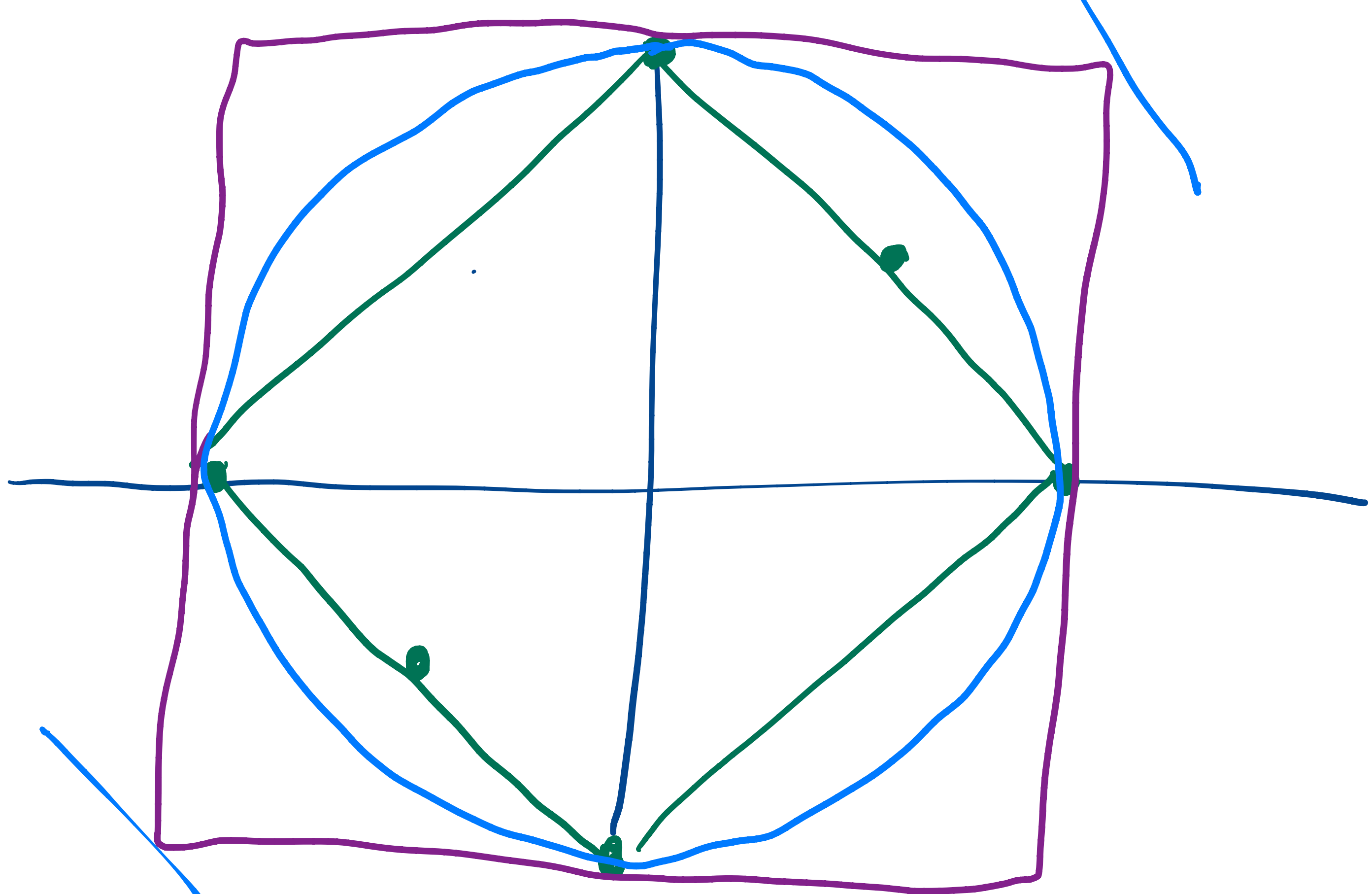
$\|x\|_1 = 1$

$\mathbf{x} = (x_1, x_2)^T$

$$||\mathbf{x}||_1 = |x_1| + |x_2|$$

$|-\frac{1}{2}| + |-\frac{1}{2}| = 1$

Unit ball: $||\mathbf{x}||_1 \leq 1$



Vector Norms

$\|x\|_2$

The 1,2 and ∞ norms are **vector norms**.

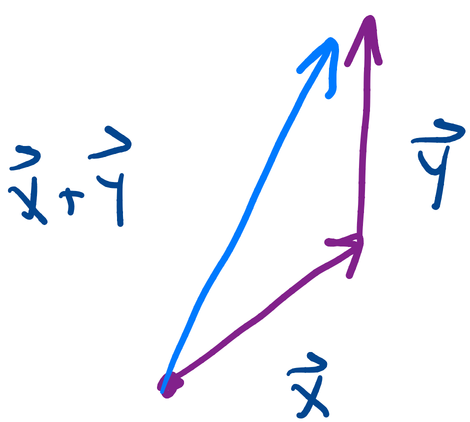
1. $\|\mathbf{x}\| \geq 0$ ($\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$)
2. $\|c\mathbf{x}\| = |c|\|\mathbf{x}\|$ for all $c \in \mathbb{R}$
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (Triangle Inequality)

$|cx_1| + |cx_2|$

$= |c|\,|x_1| + |c|\,|x_2|$

$= |c|\left(\|x\|_1\right)$

Picture of triangle inequality



Absolute and Relative Errors

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$$\mathbf{y} = \mathbf{x} + \Delta\mathbf{x}$$

$$\frac{|\Delta x|}{|x|}$$

Absolute errors:

$$||\Delta\mathbf{b}||; \qquad ||\Delta\mathbf{x}||$$

Relative errors:

$$\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}; \qquad \frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}$$

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$$\mathbf{y} = \mathbf{x} + \Delta\mathbf{x}$$

$$Ax = b$$

$$Ay = b + \Delta b$$

$$A(x + \Delta x) = b + \Delta b$$

Relationship:

$$A\Delta\mathbf{x} = \Delta\mathbf{b}$$

$$\Delta\mathbf{x} = A^{-1}\Delta\mathbf{b}$$

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Motivating Condition Number

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$$||\mathbf{b}|| = ||A\mathbf{x}||$$

Suppose

$$||A^{-1}\mathbf{y}|| \leq M||\mathbf{y}||$$
$$||A\mathbf{w}|| \leq C||\mathbf{w}||$$

no matter what $\mathbf{y}$ and $\mathbf{w}$ are.

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$$\frac{||\Delta \mathbf{x}||}{||\mathbf{x}||} \le M\frac{||\Delta \mathbf{b}||}{||\mathbf{x}||} \le M\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}\frac{||\mathbf{b}||}{||\mathbf{x}||} = M\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}\frac{||A\mathbf{x}||}{||\mathbf{x}||} \le CM\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}$$

$||A\mathbf{x}|| \le C||\mathbf{x}||$