Cubic Splines

Math 426

University of Alaska Fairbanks

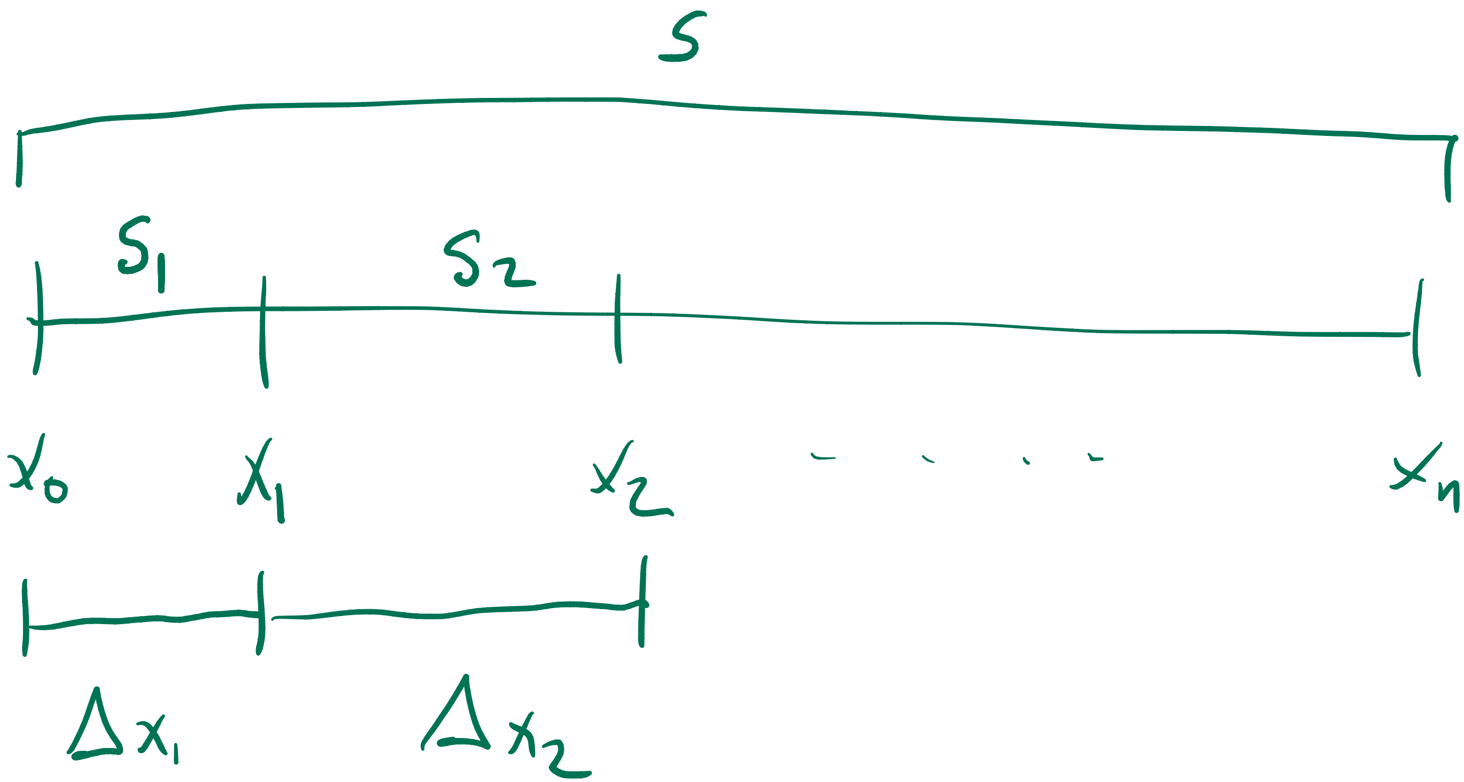
November 9, 2020

Cubic Spline Interpolation

Goal: Create a 'smooth' interpolating piecewise polynomial knowing only sample values but no sample derivatives.

Nodes: $x_0, \ldots, x_n$; values $f(x_0), \ldots, f(x_n)$.

Cubic on each interval: $4n$ unknown coefficients.



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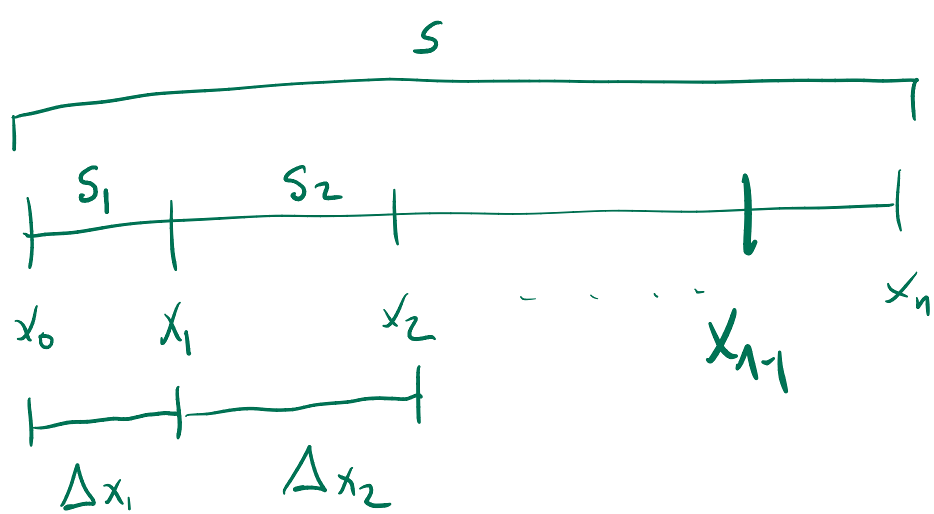

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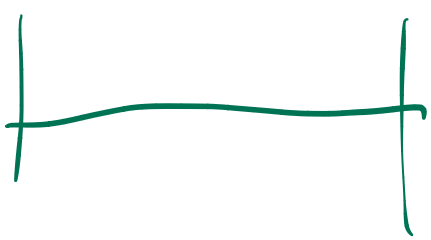

$2n + (n-1) + (n-1) = 4n - 2$

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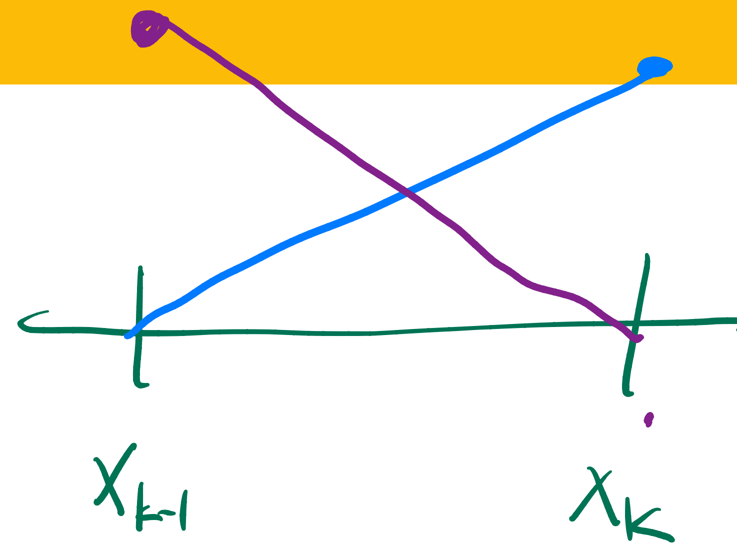
Total of $4n-2$ conditions; two more to be determined later.

Cubic Spline Derivation



Let $s(x)$ be the piecewise cubic spline.

Let $z_k = s''(x_k)$. On interval k $(1 < k < n)$:

$$s_k(x) = f(x_{k-1})(1-\theta) + f(x_k)(\theta) +$$
$$+ z_{k-1}\frac{(1-\theta)^3-(1-\theta)}{6}\Delta x_k^2 + z_k\frac{\theta^3-\theta}{6}\Delta x_k^2$$

$s_k(x_k) = f(x_k)$

$s_k(x_{k-1}) = f(x_{k-1})$

$\theta = (x - x_k)/\Delta x_k, \quad \Delta x_k = x_k - x_{k-1}$

$k-1$

This cubic matches 4 conditions (2 values, two second derivatives)

$$\theta(x) = \frac{x - x_{k-1}}{x_k - x_{k-1}} \qquad \theta(x_{k-1}) = 0 \qquad \theta(x_k) = 1$$

Cubic Spline Derivation

$$s_k''(x) = z_{k-1}(1-\theta) + z_k\theta$$

Let $s(x)$ be the piecewise cubic spline.

Let $z_k = s''(x_k)$. On interval k $(1 < k < n)$:

$$z_k \frac{3 \cdot 2 \cdot \theta - 0}{6} \frac{\Delta x_k^2}{\Delta x_k^2}$$

x_k

$$s_k(x) = f(x_{k-1})(1-\theta) + f(x_k)(\theta) +$$
$$+ z_{k-1}\frac{(1-\theta)^3 - (1-\theta)}{6}\Delta x_k^2 + z_k\frac{\theta^3 - \theta}{6}\Delta x_k^2$$

$z_k\theta$

$\theta = (x - x_{k-1})/\Delta x_k, \quad \Delta x_k = x_k - x_{k-1}$

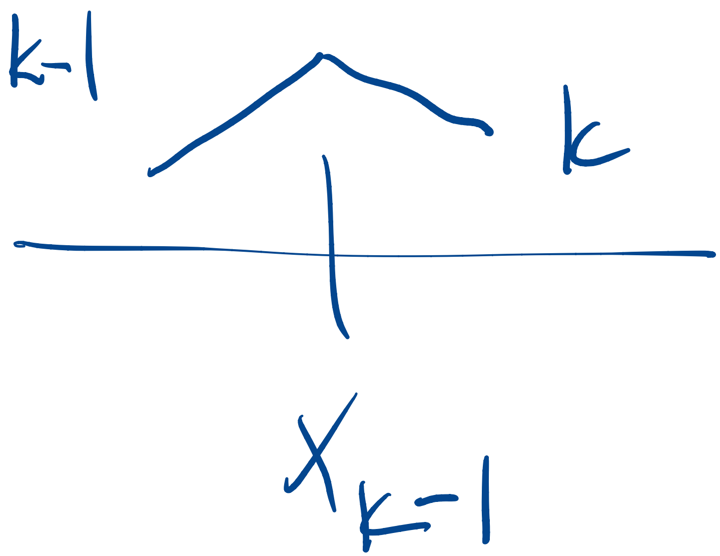
This cubic matches 4 conditions (2 values, two second derivatives)

$s_k''(x_{k-1}) = z_{k-1}$ $\qquad$ $s_k''(x_k) = z_k$

Cubic Spline Derivation

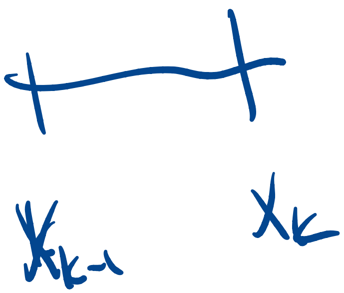
$$s_k(x) = f(x_{k-1})(1-\theta) + f(x_k)(\theta) +$$
$$+ z_{k-1}\frac{(1-\theta)^3 - (1-\theta)}{6}\Delta x_k^2 + z_k\frac{\theta^3-\theta}{6}\Delta x_k^2$$

$$s_k' = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k}$$
$$- z_{k-1}\frac{3(1-\theta)^2 - 1}{6}\Delta x_k + z_k\frac{3\theta^2-1}{6}\Delta x_k$$



Cubic Spline Derivation

$$s_k(x) = f(x_{k-1})(1-\theta) + f(x_k)(\theta) + \\ + z_{k-1}\frac{(1-\theta)^3 - (1-\theta)}{6}\Delta x_k^2 + z_k \frac{\theta^3 - \theta}{6}\Delta x_k^2$$



$$s_k' = \frac{f(x_k) - f(x_{k-1})}{\Delta x} \\ - z_{k-1}\frac{3(1-\theta)^2 - 1}{6}\Delta x_k + z_k \frac{3\theta^2 - 1}{6}\Delta x_k$$

$$x_{k-1} \leftrightarrow \theta = 0$$

$$s_k'(x_{k-1}) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} - z_{k-1}\frac{1}{3}\Delta x_k - \frac{1}{6} z_k \Delta x_k$$

Cubic Spline Derivation

$$s_k(x) = f(x_{k-1})(1-\theta) + f(x_k)(\theta) +$$
$$+ z_{k-1}\frac{(1-\theta)^3 - (1-\theta)}{6}\Delta x_k^2 + z_k\frac{\theta^3 - \theta}{6}\Delta x_k^2$$

$$s_k' = \frac{f(x_k) - f(x_{k-1})}{\Delta x}$$
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$$x_{k-1} \leftrightarrow \theta = 0$$

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$$x_k \leftrightarrow \theta = 1$$

$$s_k'(x_{k-1}) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} + z_k\frac{1}{3}\Delta x_k + \frac{1}{6}z_{k-1}\Delta x_k$$

First Derivative Matching

$$s'_k(x_{k-1}) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} - z_{k-1}\frac{1}{3}\Delta x_k - \frac{1}{6}z_k \Delta x_k$$

$$s'_k(x_k) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} + z_k\frac{1}{3}\Delta x_k + \frac{1}{6}z_{k-1} \Delta x_k$$

Want, $2 \leq k \leq n-1$,

$$s'_{k-1}(x_{k-1}) = s'_k(x_{k-1})$$

$$\frac{\Delta f_{k-1}}{\Delta x_{k-1}} + \frac{1}{6}z_{k-2}\Delta x_{k-1} + \frac{1}{3}z_{k-1}\Delta x_{k-1} = \frac{\Delta f_k}{\Delta x_k} - \frac{1}{3}z_{k-1}\Delta x_k - \frac{1}{6}z_k \Delta x_k$$

$$\Delta f_k = f(x_k) - f(x_{k-1})$$

First Derivative Matching

$$s_k'(x_{k-1}) = \frac{f(x_k) - f(x_{k-1})}{\Delta x_k} - z_{k-1}\frac{1}{3}\Delta x_k - \frac{1}{6}z_k\Delta x_k$$

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Want, $2 \leq k \leq n-1$,

$$s_{k-1}'(x_{k-1}) = s_k'(x_{k-1})$$

$$\frac{\Delta f_{k-1}}{\Delta x_{k-1}} + \frac{1}{6}z_{k-2}\Delta x_{k-1} + \frac{1}{3}z_{k-1}\Delta x_{k-1} = \frac{\Delta f_k}{\Delta x_k} - \frac{1}{3}z_{k-1}\Delta x_k - \frac{1}{6}z_k\Delta x_k$$

$$\frac{1}{6}\Delta x_{k-1}z_{k-2} + \frac{1}{3}(\Delta x_{k-1} + \Delta x_k)z_{k-1} + \frac{1}{6}\Delta x_k z_k = \frac{\Delta f_k}{\Delta x_k} - \frac{\Delta f_{k-1}}{\Delta x_{k-1}} := \Delta\Delta f_k$$

Cubic Spline Interpolation

$2n \quad \cancel{+\, n-1 = 3n-1}$

$\to 4n$

For $2 \le k \le n-1$

$S''_{k-1}(x_{k-1}) = z_{k-1}$

$S''_k(x_{k-1}) = z_{k-1}$

$$\boxed{\beta_k = \frac{\Delta x_k}{6}; \quad \alpha_{k-1} = \frac{1}{3}(\Delta x_{k-1} - \Delta x_k)}$$

$$\cancel{\beta_{k-1} z_{k-2} + \alpha_{k-1} z_k + \beta_k z_{k+1} = \Delta\Delta f_k}$$

whoops!

$$\beta_{k-1} z_{k-2} + \alpha_{k-1} z_{k-1} + \beta_k z_k = \Delta\Delta f_{k-1}$$

$$\begin{pmatrix} * & \cdots & \cdots & \cdots & \cdots & * \\ \beta_1 & \alpha_1 & \beta_2 & 0 & 0 & \cdots \\ 0 & \beta_2 & \alpha_2 & \beta_3 & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ * & \cdots & \cdots & \cdots & \cdots & * \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} * \\ \Delta\Delta f_1 \\ \vdots \\ \Delta\Delta f_{n-1} \\ * \end{pmatrix}$$

3:30 – 6:00

8:30 – 9

- natural $\overset{=0}{z_0}, \overset{=0}{z_n}$

$$s''(x_0)=0 \qquad s''(x_n)=0$$

- complete $s'(x_0) \qquad s'(x_n)$

- not a knot

s''' exists at x_1 and x_{n-1}