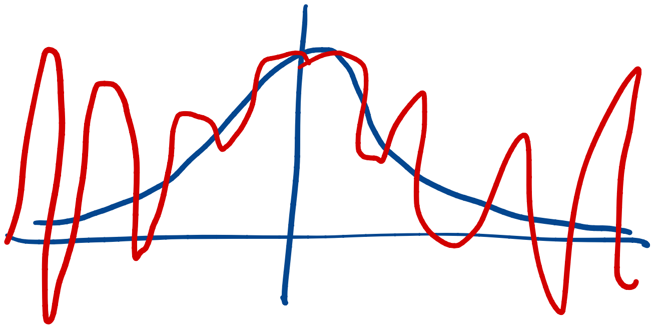
Piecewise Polynomial Interpolation

Math 426

University of Alaska Fairbanks

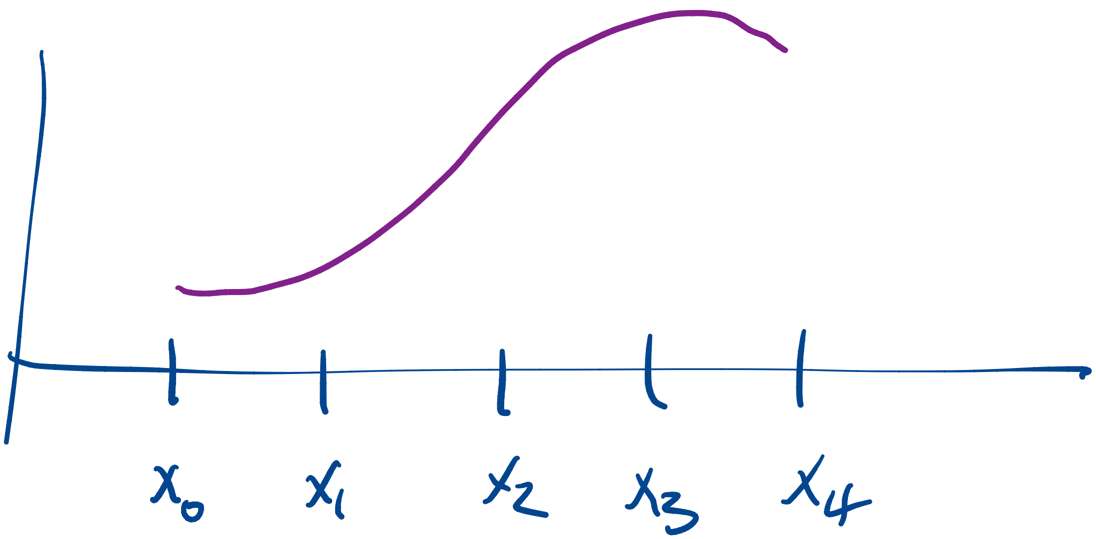
November 6, 2020

Motivation



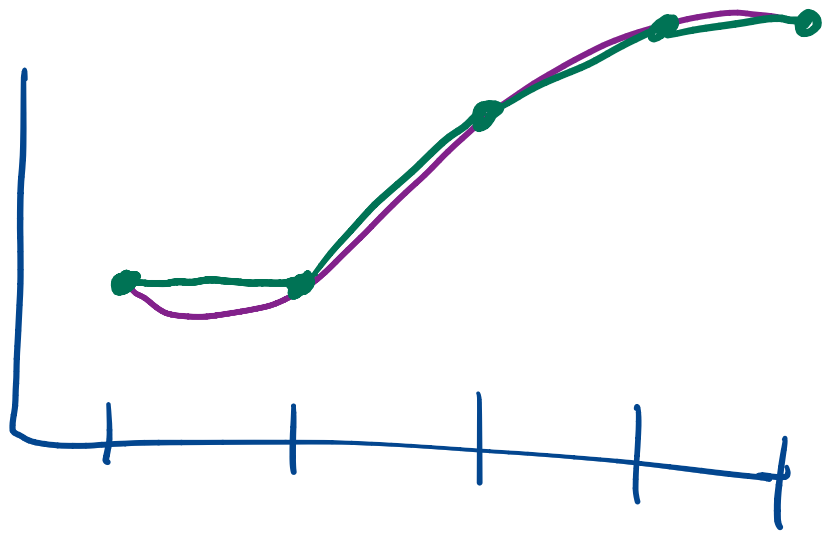
High order polynomials wiggle a lot. Your function might not be very wiggly.

An alternative to using a high order polynomial to approximate the function on the whole domain is to use many low order polynomials to approximate the function on small pieces of the domain.



Piecewise Linear Interpolation

Diagram:



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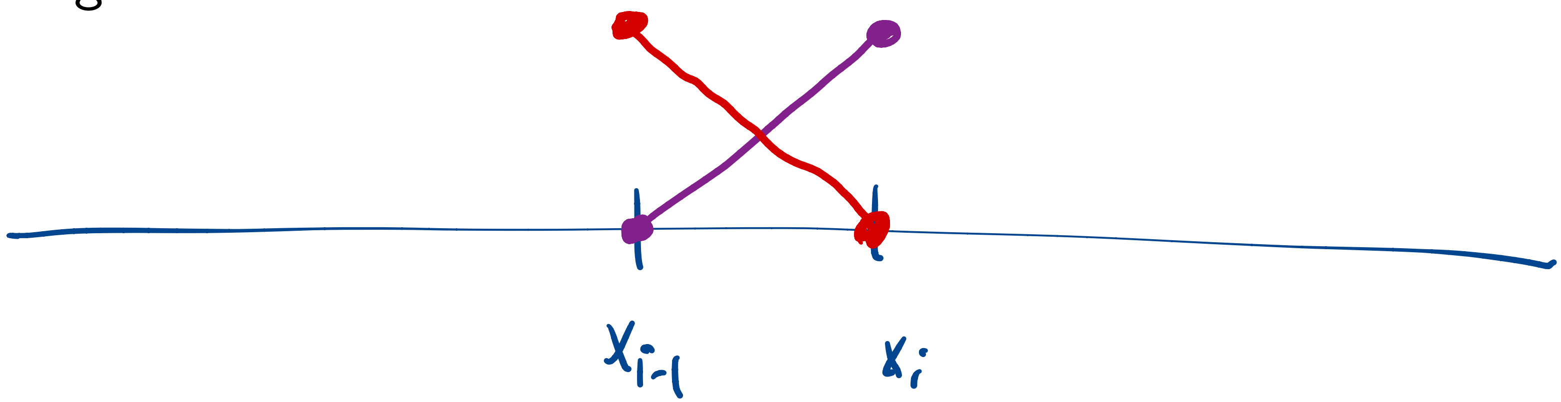

Handy notation: on interval i,

$$\theta_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}.$$

$\theta_i(x_{i-1}) = 0$

$\theta_i(x_i) = 1$

$1 - \theta_i(x)$

Piecewise Linear Interpolation

Diagram:

$p_i(x_{i-1}) = f(x_{i-1})$

$p_i(x_i) = f(x_i)$

Handy notation: on interval i,

$$\theta_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}.$$

On each interval,

$$p_i(x) = f(\cancel{x_i - 1})(1 - \theta_i(x)) + f(x_i)\theta_i(x)$$

x_{i-1}

Error in Piecewise Linear Interpolation

On each subinterval, we can use the error from standard polynomial interpolation:

$$f(x) = p(x) + \frac{f''(\xi)}{2}(x - x_{i-1})(x - x_i)$$

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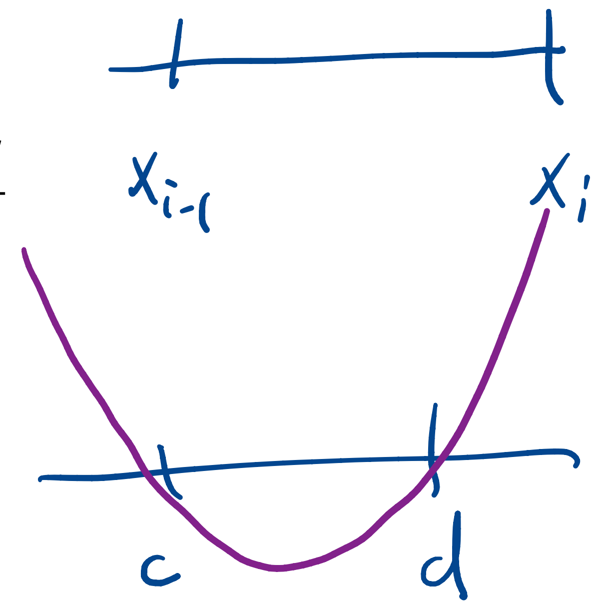
: Exercise:

$$|(x-c)(x-d)| \leq \frac{|d-c|^2}{2}$$



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$x_i - x_{i-1} = \Delta x_i$

: Exercise:

$$|(x-c)(x-d)| \leq \frac{|d-c|^2}{2}$$

If $|f''(\xi)| \leq M$ everywhere

$$|f(x) - p(x)| \leq \frac{M}{8} \max(\Delta x_i)^2$$

4

n

$\Delta x_i = \Delta x = \frac{x_n - x_0}{n}$

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If $\Delta x_i = (b - a)/n$,

$$|f(x) - p(x)| \le Cn^{-2}$$

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If $\Delta x_i = (b - a)/n$,

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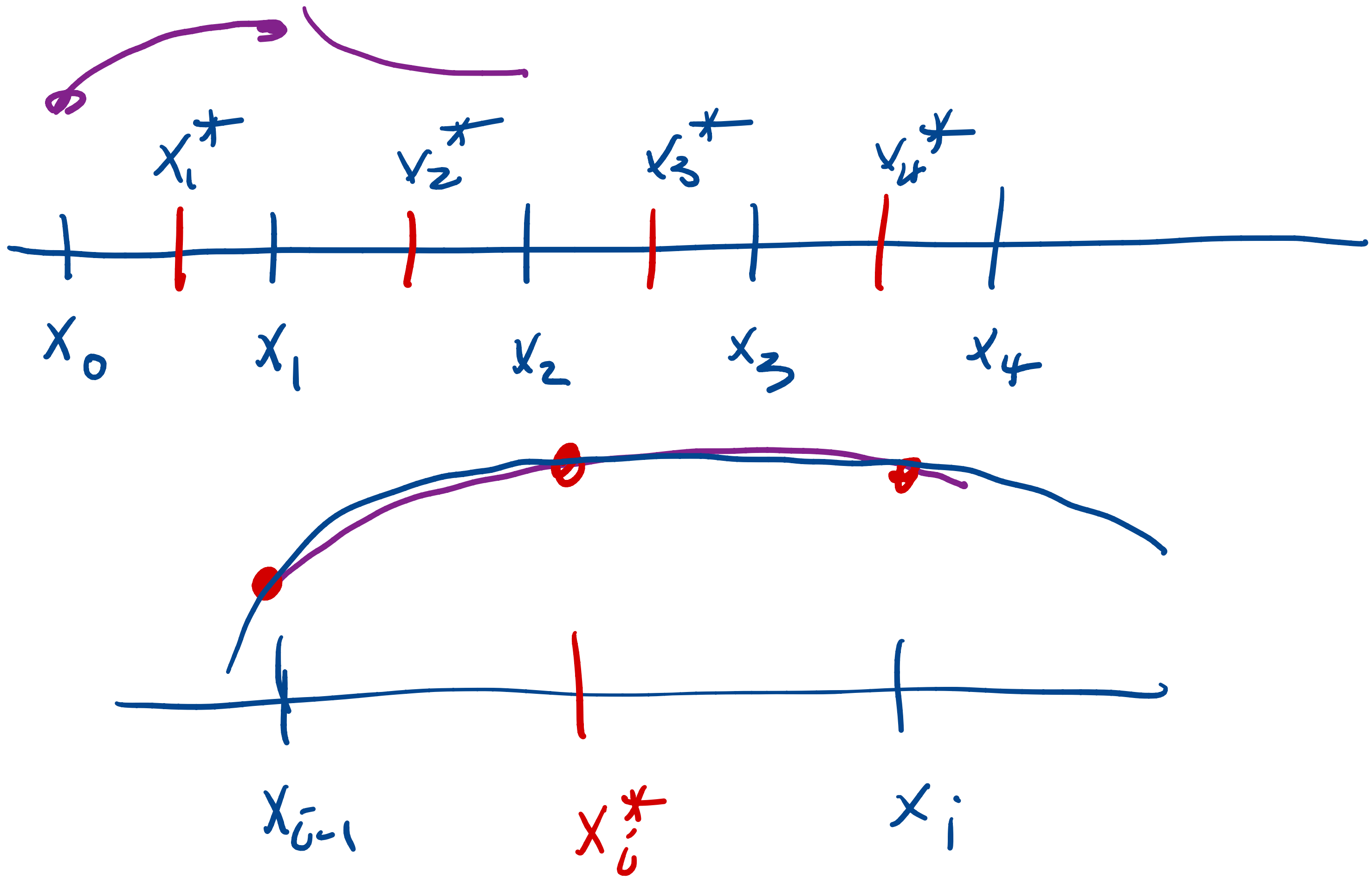
As good as Chebyshev for a twice differentiable (only) function.

Piecewise Quadratic Interpolation

Interval endpoints: $x_0, x_1, \ldots x_n$. Midpoints:

$$x_k^* = \frac{x_k + x_{k-1}}{2}; 1 \leq k \leq n.$$



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Fit a quadratic polynomial at x_{k-1}, x_k^*, x_k.

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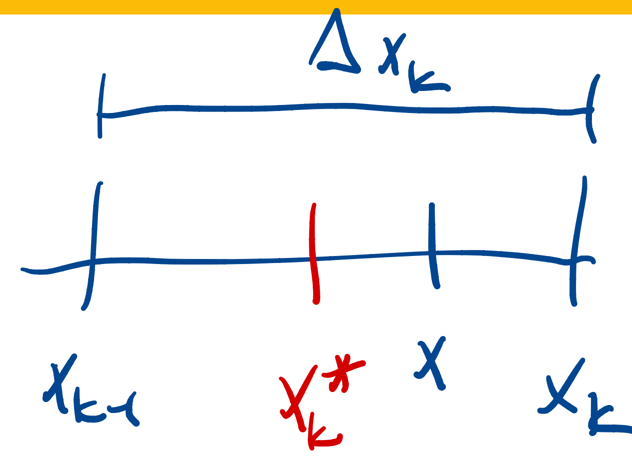

Fit a quadratic polynomial at x_{k-1}, x_k^*, x_k.

On interval ~~i~~ k.

$$f(x) = p_k(x) + \frac{f^{(3)}(\xi)}{3!}\underbrace{(x - x_{k-1})}_{\Delta x_k}\underbrace{(x - x_k^*)}_{\frac{\Delta x_k}{2}}\underbrace{(x - x_k)}_{\Delta x_k}$$

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$$f(x) = p_k(x) + \frac{f^{(3)}(\xi)}{3!}(x - x_{k-1})(x - x_k^*)(x - x_k)$$

$$|f(x) - p_k(x)| \leq C(\Delta x_k)^3 \leq C' n^{-3}$$

$$\Delta x_k = \frac{b-a}{n}$$

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If $\Delta x_i = (b - a)/n$,

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for all x.

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If $\Delta x_i = (b-a)/n$,

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for all x.

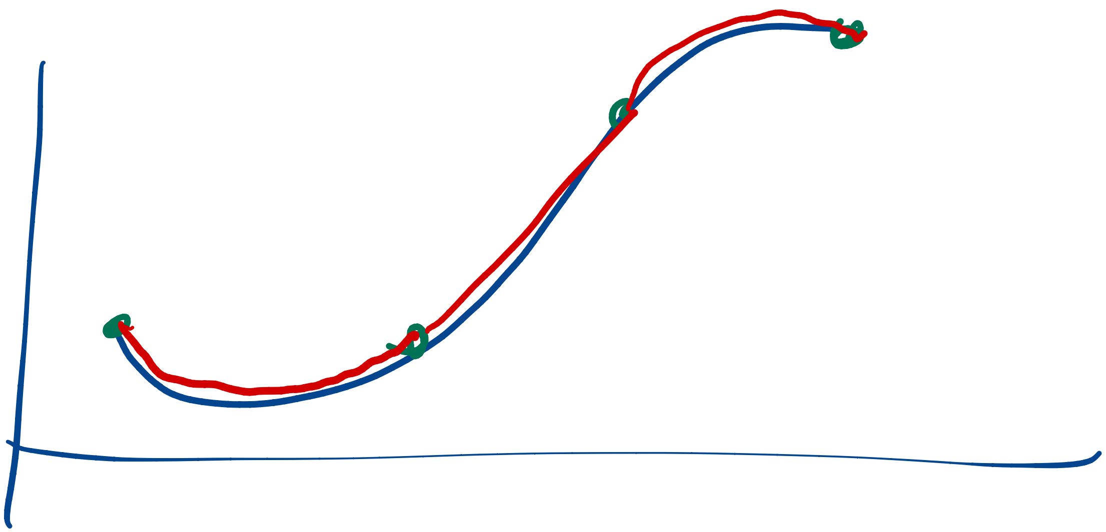
Same kind of estimate as Chebyshev for a three times (only) differentiable f.

Interpolation and Calculus

Integrating and differentiating polynomials is easy.

We can use piecewise interpolating polynomials to estimate definite integrals easily: just integrate the polynomials on each interval and add 'em up.

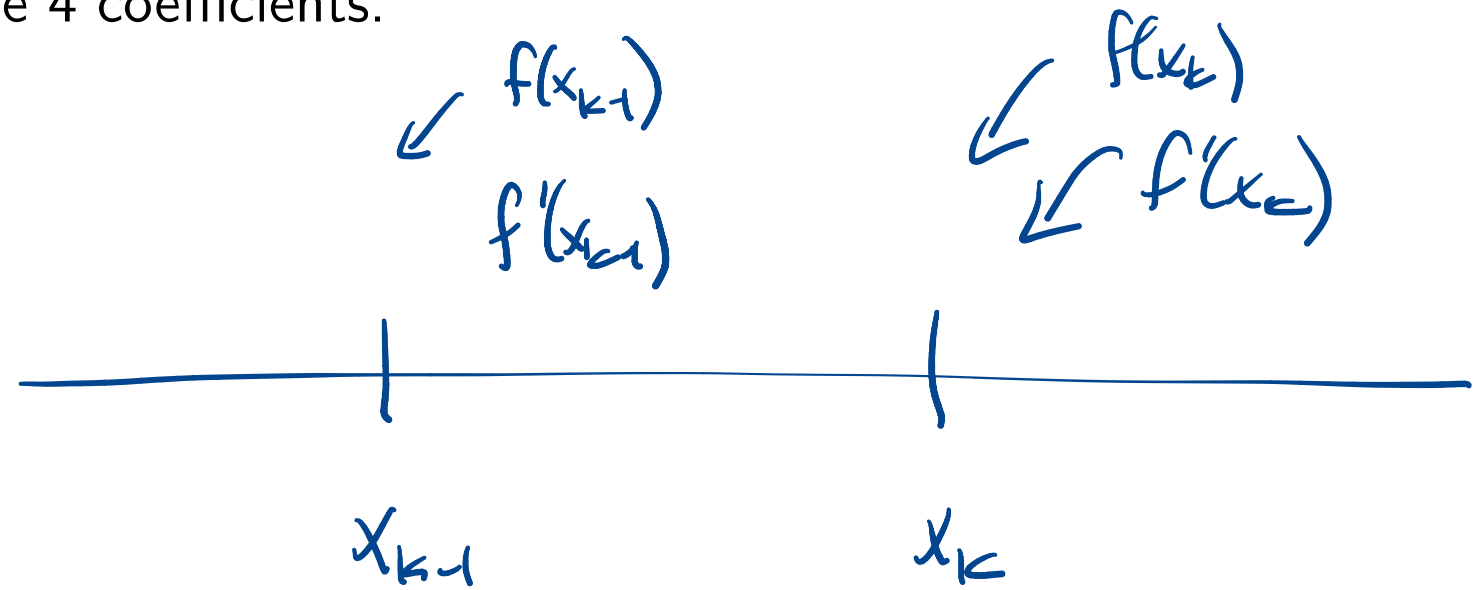
Derivatives are more complicated: there is no reason that derivatives should match up at the interval endpoints:



Cubic Hermite Interpolation

Suppose you know both $f(x)$ and $f'(x)$ at $x_0, x_1, \ldots, x_n$.

On each $[x_{i-1}, x_i]$ this yields 4 conditions, and cubic polynomials have 4 coefficients.



$c_2 x^2 + c_1 x + c_0$

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p' will be a quadratic with $p'(x_{i-1}) = f'(x_{i-1})$ and $p'(x_i) = f'(x_i)$.

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$$p'(x) = f'(x_{i-1})(1 - \theta_i(x)) + f'(x_i)\theta_i(x) + \alpha(x - x_{i-1})(x - x_i).$$

$$p(\overset{x}{\cancel{x_i}}) = p(x_{i-1}) + \int_{x_{i-1}}^{\overset{x}{\cancel{x_i}}} p'(s)\, ds$$

$$p(x) = f(x_{i-1}) + \Delta x_i \left[\theta f'(x_{i-1}) + \frac{\theta^2}{2}(f'(x_i) - f'(x_{i-1})) + \alpha \frac{\theta^2}{6}(\theta^2)(3 - 2\theta) \right]$$

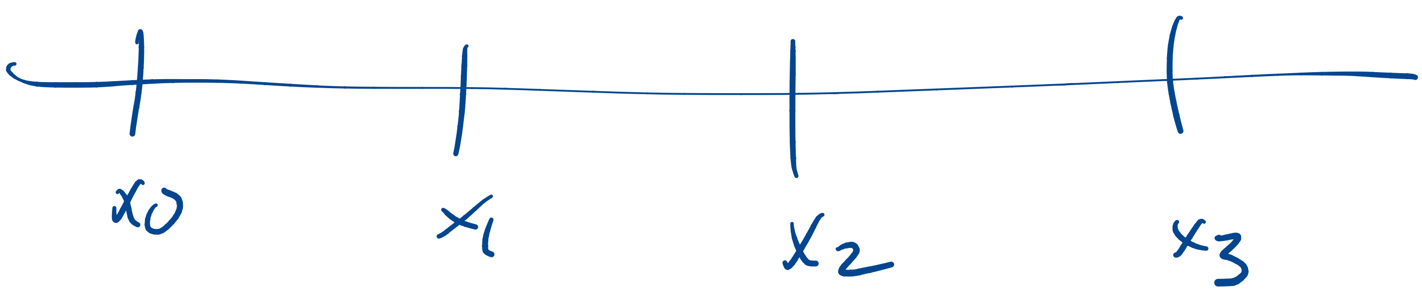
$$\alpha = 6 \left[\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} - (f'(x_{i+1}) + f'(x_i))/2 \right]$$

Cubic Spline Interpolation

Goal: Create a 'smooth' interpolating piecewise polynomial knowing only sample values but no sample derivatives.

Nodes: $x_0, dots x_n$; values $f(x_0), \ldots, f(x_n)$.

Cubic on each interval: $4n$ unknown coefficients.



$2n$
$n-1$
$n-1$

$4n-2$