Error in Linear/Quadratic Interpolation

Linear case: Error should depend on

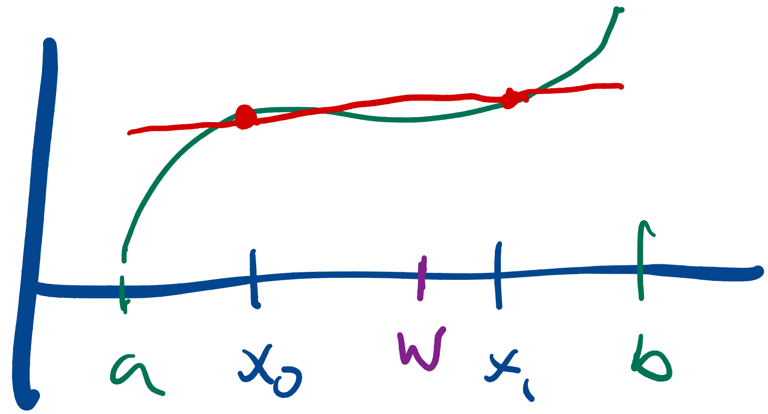
1. $x - x_0$
2. $x - x_1$
3. something about f''

Quadratic case: Error should depend on

1. $x - x_0$
2. $x - x_1$
3. $x - x_2$
4. something about f'''

Maybe something like $\max|f'''(\xi)||(x - x_0)(x - x_1)(x - x_2)|$??

Error in Linear Interpolation (Exactly!)



Interpolation points: x_0, x_1.

$p(x)$ linear, $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$.

Want to estimate $p(w) - f(w)$ for some w in $[a, b]$.

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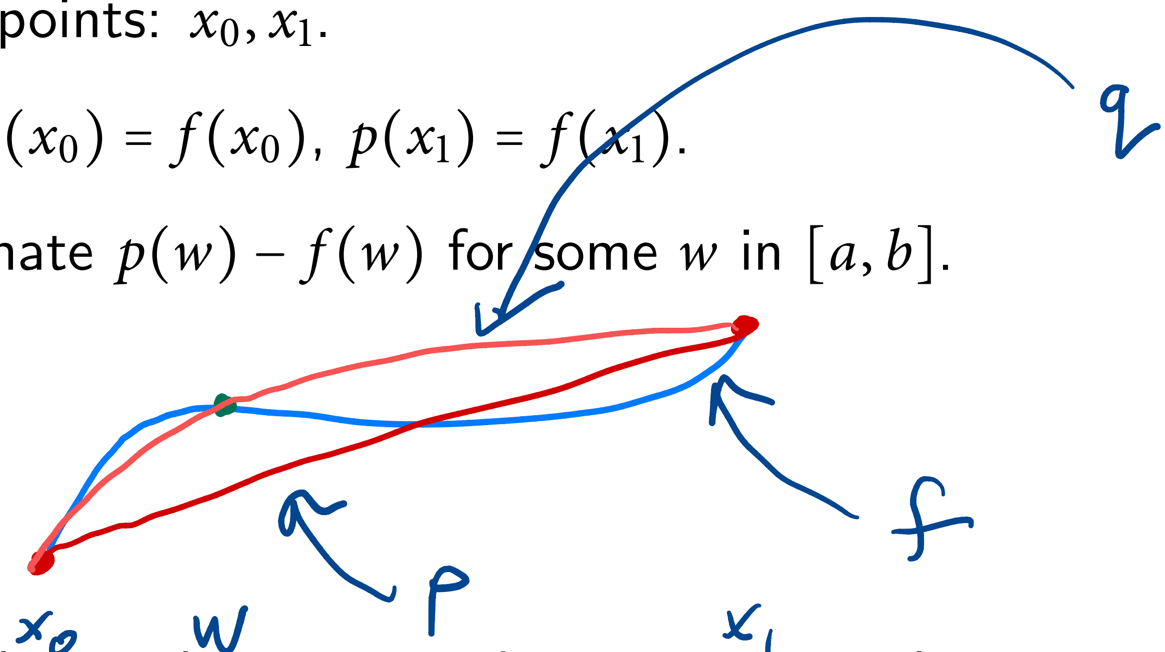

Let $q(x)$ be the quadratic interpolant at x_0, x_1 and at w.

Error in Linear Interpolation (II)

Interpolation points: x_0, x_1.

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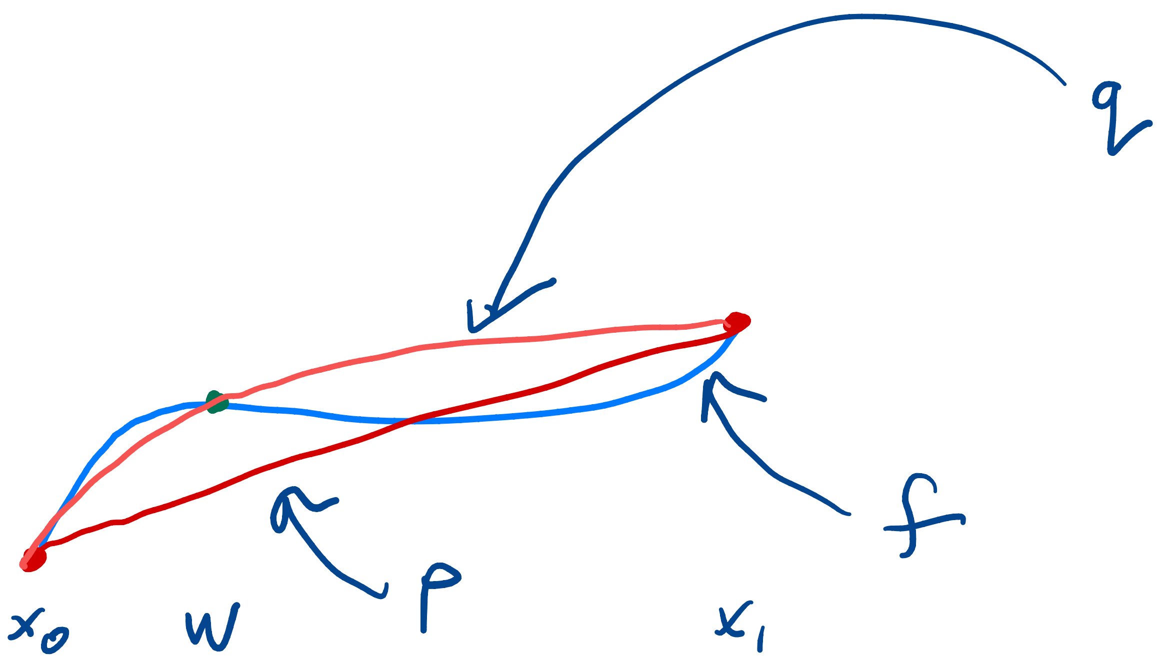
$q(x)$ quadratic, $q(x_0) = f(x_0)$, $q(w) = f(w)$, $q(x_1) = f(x_1)$



Error in Linear Interpolation (II)

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$q(x)$ quadratic, $q(x_0) = f(x_0)$, $q(w) = f(w)$, $q(x_1) = f(x_1)$

$$q(x) = p(x) + \lambda(x - x_0)(x - x_1)$$

$$q(x_0) = p(x_0) + \lambda(x_0 - x_0)(x_0 - x_1)$$

$$= p(x_0) = f(x_0)$$

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$$q(x) = p(x) + \lambda(x - x_0)(x - x_1)$$

$$\lambda = \frac{f(w) - p(w)}{(w - x_0)(w - x_1)}$$

$$q(w) = f(w)$$

$$q(w) = p(w) + \frac{(f(w) - p(w))}{(w - x_0)(w - x_1)}(w - x_0)(w - x_1)$$

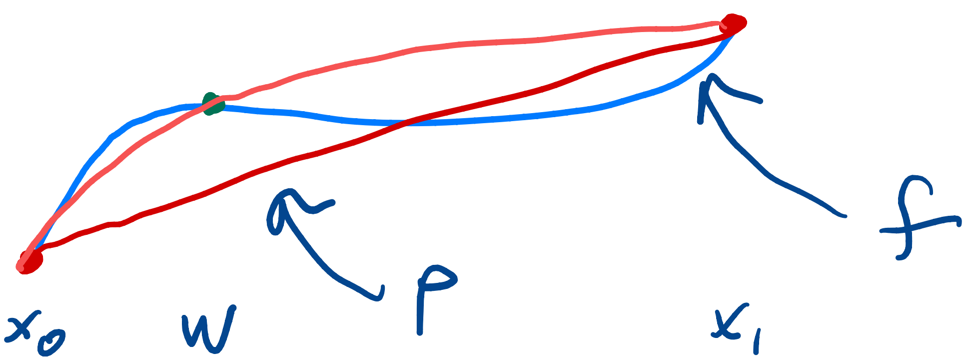
Error in Linear Interpolation (III)

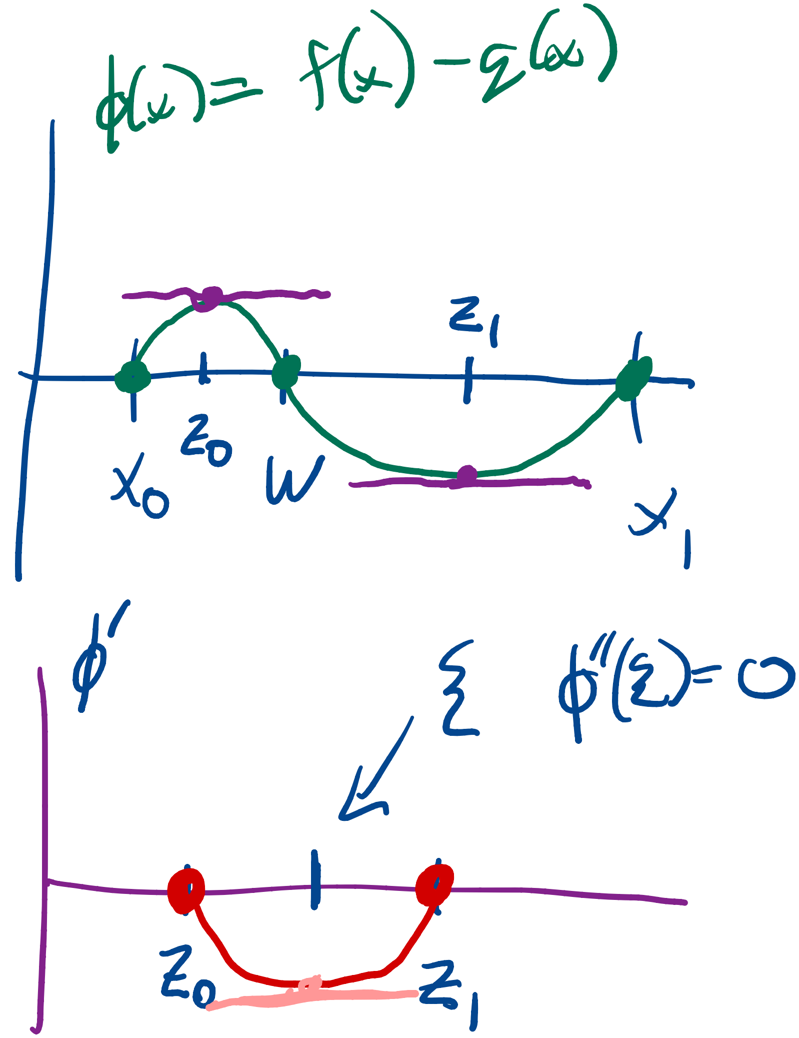
$q(x)$ quadratic, $q(x_0) = f(x_0)$, $q(w) = f(w)$, $q(x_1) = f(x_1)$

$$q(x) = p(x) + \lambda(x - x_0)(x - x_1)$$

$$\lambda = \frac{f(w) - p(w)}{(w - x_0)(w - x_1)}$$

Let's graph $\phi(x) = f(x) - q(x)$:





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Let's graph $\phi(x) = f(x) - q(x)$:

There are two points z_0 and z_1 such that $\phi'(x) = 0$

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Mean Value Theorem sez:

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Mean Value Theorem sez: $\phi''(\xi) = 0$ at some point between z_0 and z_1.

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$$q(x) = p(x) + \lambda(x - x_0)(x - x_1)$$

x^2 + linear crap

$$\lambda = \frac{f(w) - p(w)}{(w - x_0)(w - x_1)}$$

$\phi(x) = f(x) - q(x)$

$$\phi''(\xi) = 0$$

somewhere.

$$0 = \phi''(\xi) = f''(\xi) - q''(\xi) = f''(\xi) - 2\lambda$$

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$\phi(x) = f(x) - q(x)$

$$\phi''(\xi) = 0$$

somewhere.

$$\phi''(\xi) = f''(\xi) - 2\lambda = f''(\xi) - 2\frac{f(w) - p(w)}{(w - x_0)(w - x_1)}$$

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$\phi(x) = f(x) - q(x)$

$$\phi''(\xi) = 0$$

somewhere.

$$\phi''(\xi) = f''(\xi) - 2\lambda = f''(\xi) - 2\frac{f(w) - p(w)}{(w - x_0)(w - x_1)}$$

$$f(w) = p(w) + f''(\xi)\frac{(w - x_0)(w - x_1)}{2!}.$$

General Interpolation Error

Theorem

Suppose f is $n+1$ times differentiable on $[a,b]$ and $x_0,\ldots,x_n \in [a,b]$. Let p be the polynomial interpolant of f at these points. Then for all $x \in [a,b]$ there exists $\xi \in [a,b]$ such that

$$f(x) = p(x) + f^{(n+1)}(\xi)\frac{\prod_{k=0}^{n}(x-x_k)}{(n+1)!}.$$

$n=1$

$$f(x) = p(x) + \frac{f''(\xi)}{2!}(x-x_0)(x-x_1)$$

$w \longleftrightarrow x$