Derivative Approximation

Math 426

University of Alaska Fairbanks

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The basic approximation

Recall:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

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Recall:

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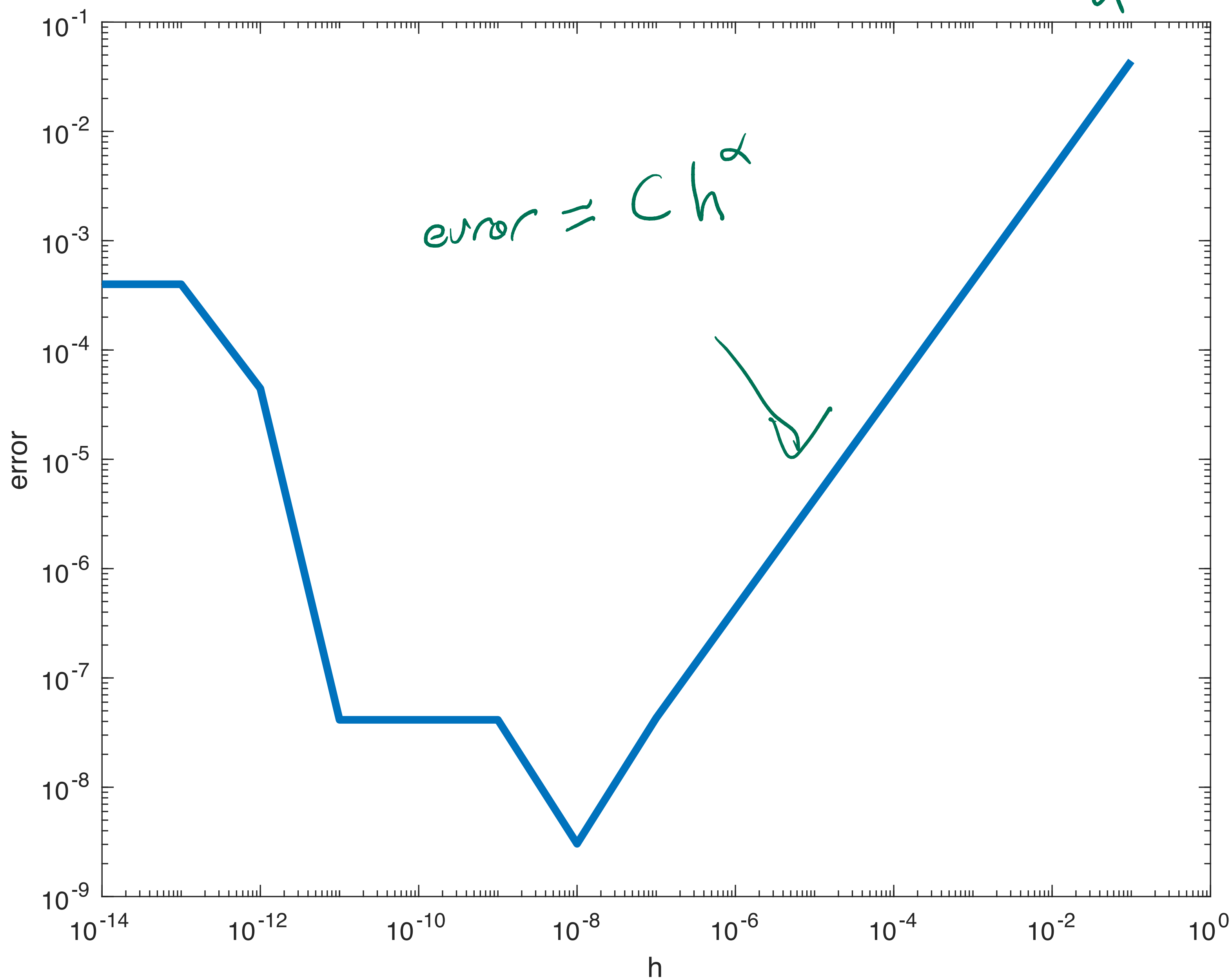
So for small values of h,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

In practice

Approximation of $\sin'(\pi/3)$.

$$\frac{\sin(\pi/3+h)-\sin(\pi/3)}{h} - \cos\left(\frac{\pi}{3}\right)$$



Two things to explain

1. What is the slope for larger values of h?
2. Why to things break down for smaller values of h?

Taylor Approximation

$$f(x+h) = f(x) + f'(x)h + f''(\xi)h^2/2$$

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$$\frac{f(x+h) - f(x)}{h} = f'(x) + f''(\xi)\frac{h}{2}$$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + O(h)$$

$O(h^2)$

Taylor Approximation

More specifically:

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2} + f'''(\xi)\frac{h^3}{3}$$

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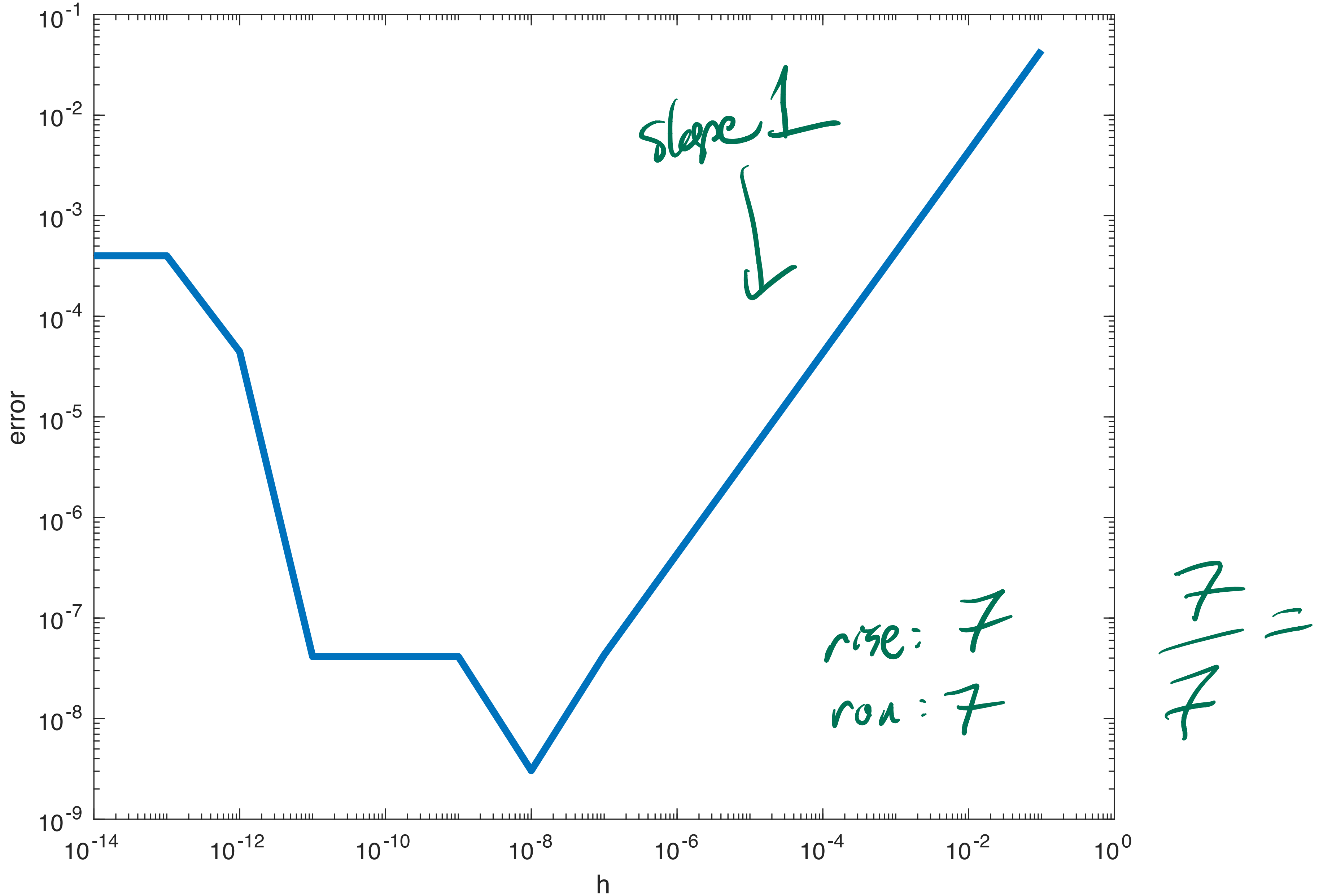
$$\frac{f(x+h) - f(x)}{h} = f'(x) + f''(x)\frac{h}{2} + f'''(\xi)\frac{h^3}{3}$$

In practice

Line has slope 1, and error is $O(h^1)$.



Rounding error

We don't really compute the difference quotient:

$$\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h}$$

with $\delta \approx \epsilon = 10^{-16}$.

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$$\left(\frac{f(x+h) - f(x)}{h} - f'(x) \right) + \frac{f(x+h)\delta_1 - f(x)\delta_2}{h}$$

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$$\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} - f'(x) \sim f''(x)h/2 + \frac{2f(x)}{h}\epsilon$$

Minimize error

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$g(h) = C_1 h + \frac{C_2 \varepsilon}{h}$

$$g(h) = C_1 h + C_2\epsilon/h g'(h) = C_1 - C_2\epsilon/h^2$$

Minimum requires $g'(h) = 0$,

10^{-16}

$$h = \sqrt{\epsilon}\sqrt{\frac{C_2}{C_1}} \sim 10^{-8}$$

$$g(h) = C_1 h + \frac{C_2 \varepsilon}{h}$$

Minimize: $g'(h) = 0$

$$g'(h) = C_1 - \frac{C_2 \varepsilon}{h^2}$$

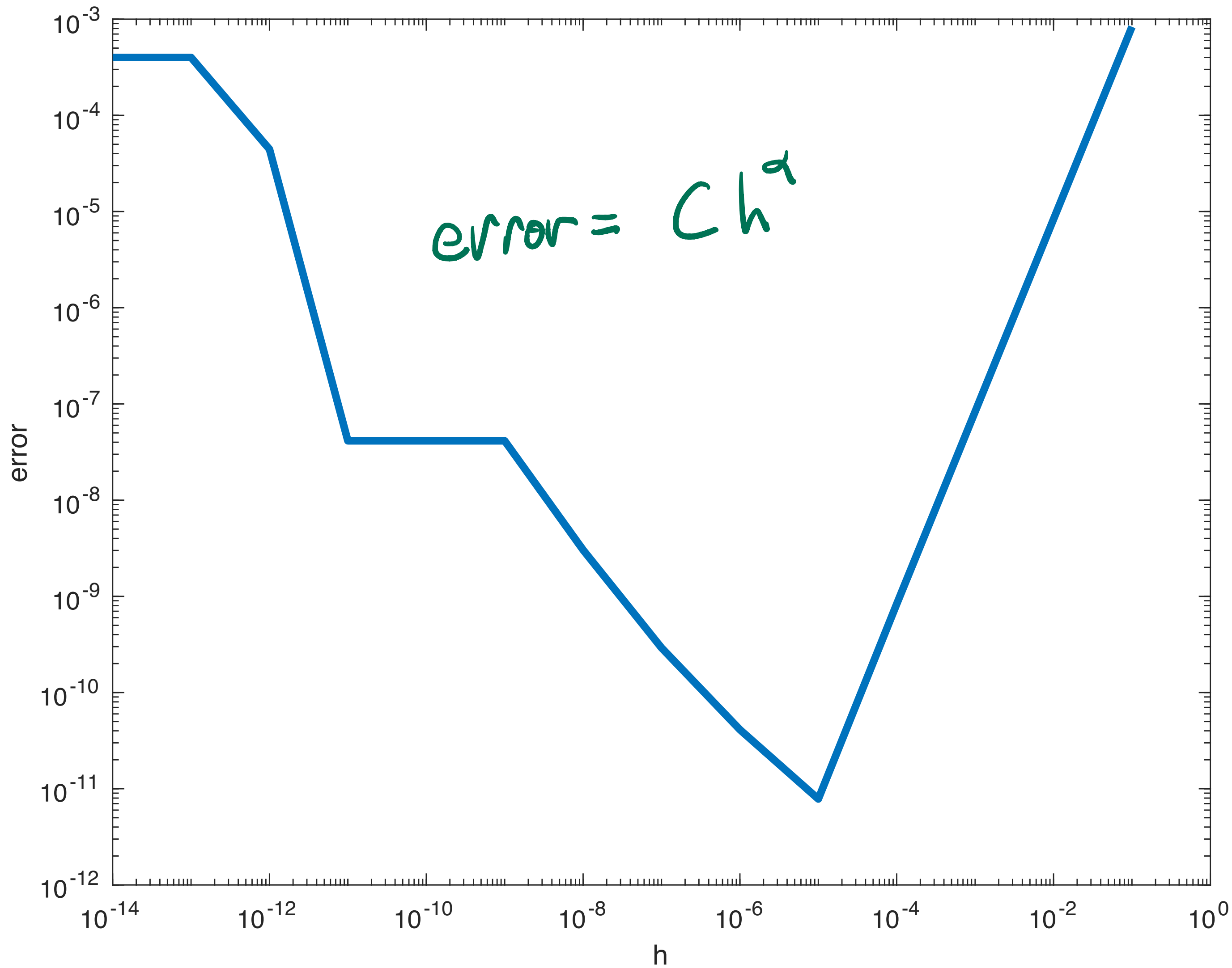
$$C_1 - \frac{C_2 \varepsilon}{h^2} = 0 \qquad h^2 = \frac{C_2 \varepsilon}{C_1}$$

Centered differences

We also looked at

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Error is $O(h^2)$



Centered differences

The centered difference approximation is $O(h^2)$.

10^{-16}

$$\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} - f'(x) \sim C_1 h^2 + \frac{C_2}{h}\epsilon$$

$$g(h) = C_1 h^2 + \frac{C_2}{h}\varepsilon$$

$$g'(h) = 0$$

$$g'(h) = 2C_1 h - \frac{C_2 \varepsilon}{h^2}$$

$$h = C_3 \varepsilon^{1/3} \approx 10^{-5}$$

$$2C_1 h^3 = C_2 \varepsilon$$

$$h^3 = \frac{C_2}{2C_1}\varepsilon$$

Centered differences

The centered difference approximation is $O(h^2)$.

$$\frac{f(x+h)(1+\delta_1) - f(x)(1+\delta_2)}{h} - f'(x) \sim C_1 h^2 + \frac{C_2}{h}\epsilon$$

Now minimize at h with

$$2C_1 h - \frac{C_2}{h^2}\epsilon = 0$$

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$$h = \sqrt[3]{\epsilon}\left(\frac{C_2}{2C_1}\right)^{\frac{1}{3}} \sim 10^{-5}$$

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