Polynomial Interpolation Error

Math 426

University of Alaska Fairbanks

November 2, 2020

Mean Value Theorem

Theorem

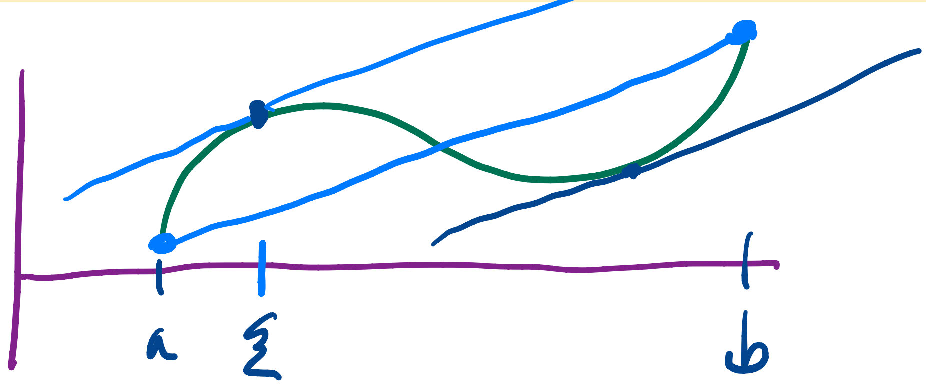
(a, b)

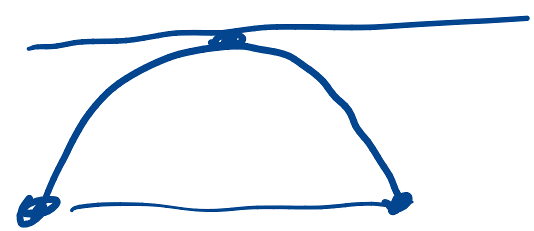
If f differentiable on $[a, b]$ there is a point ξ where

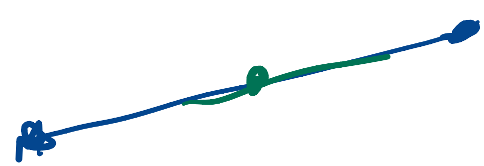
$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

or equivalently

$$f(b) = f(a) + f'(\xi)(b - a).$$







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Taylor's Theorem (0^{th} order)

AKA: The Mean Value Theorem

$[a, x]$

old b

Theorem

If f differentiable on $[a, b]$ then for every x in (a, b) there is a point ξ between a and x such that

$$f(x) = f(a) + f'(\xi)(x - a).$$

Taylor's Theorem (first order)

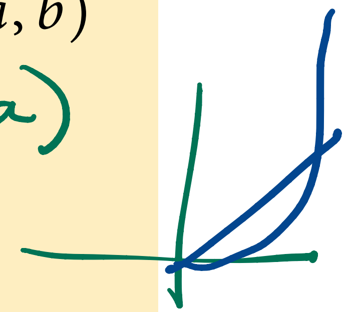
$x \to a$

Theorem

If f is two times differentiable on $[a, b]$ then for every x in (a, b) there is a point ξ between a and x such that

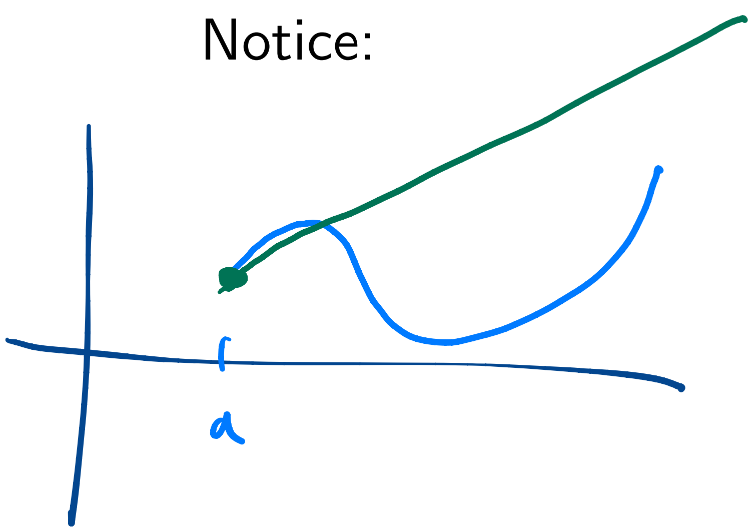
$(x-a)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2!}(x-a)^2.$$



Taylor polynomal: $p(x) = f(a) + f'(a)(x-a)$.

Notice:



$$p(a) = f(a)$$
$$p'(a) = f'(a)$$

$p'(x) = 0 + f'(a) \cdot ($

$p'(x) = f'(a)$

everywhere.

Taylor's Theorem (second order)

Theorem

If f is three times differentiable on $[a, b]$ there is a point ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(\xi)}{3!}(x-a)^3.$$

Taylor polynomal: $p(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2/2!$

Error term $f'''(\xi)/3!(x-a)^3 = f(x) - p(x)$.

$p'(x) = f'(a) + f''(a)(x-a)$

Notice:

$$p(a) = f(a)$$
$$p'(a) = f'(a)$$
$$p''(a) = f''(a)$$

$p''(x) = f''(a)$

$$f(x) = \overbrace{f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}^{p(x)}$$

$$+ \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

$\to$ error term.

$$\left| f(x) - p(x) \right| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \right|$$

Eg: Error in Cosine

$f(x) = \cos(x)$ $a = 0$ 6^{th} order Talor poly.

Derivatives:

$$
\begin{aligned}
f(0) &= \cos(0) = 1 \\
f'(0) &= -\sin(0) = 0 \\
f''(0) &= -\cos(0) = -1 \\
f'''(0) &= \sin(0) = 0 \\
f^4(0) &= 1. \\
f^5(0) &= 0 \\
f^6(0) &= -1
\end{aligned}
$$

$$p(x) = 1 - \frac{1}{2!}(x-0)^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Eg: Error in Cosine

$f(x) = \cos(x)$

Sixth order Taylor polynomial:

$$p(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

$$\frac{f^{(7)}(\xi)}{7!}(x-0)^7$$

error

$$\left|\frac{\sin(\xi)}{7!}x^7\right| \leq \frac{|x^7|}{7!}$$

Eg: Error in Cosine

$f(x) = \cos(x)$

Sixth order Taylor polynomial:

$$p(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Error term:

$$\frac{1}{7!}\sin(\xi)x^7$$

Eg: Error in Cosine

$f(x) = \cos(x)$

Sixth order Taylor polynomial:

$$p(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

$\pi^7/7!$

Error term:

$$\frac{1}{7!}\sin(\xi)x^7$$

$\frac{|x|^7}{7!}$

Maximum error $|f(x) - p(x)|$ on $[-\pi, pi]$?

$[-\pi, \pi]$?

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$f(x) = \cos(x)$

Sixth order Taylor polynomial:

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Maximum error $|f(x) - p(x)|$ on $[-\pi, pi]$? $(\pi)^7/7! \approx 0.6$

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Error term:

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Maximum error $|f(x) - p(x)|$ on $[-\pi, pi]$? $(\pi)^7/7! \approx 0.6$Maximum error $|f(x) - p(x)|$ on $[-\pi/2, pi/2]$?

$[-\frac{\pi}{2}, \pi/2]$. $(\pi/2)^7/7!$.

Eg: Error in Cosine

$f(x) = \cos(x)$

Sixth order Taylor polynomial:

$$p(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Error term:

$$\frac{1}{7!}\sin(\xi)x^7$$

Maximum error $|f(x) - p(x)|$ on $[-\pi, pi]$? $(\pi)^7/7! \approx 0.6$Maximum error $|f(x) - p(x)|$ on $[-\pi/2, pi/2]$? $(\pi/2)^7/7! \approx 0.005$

Moral of Taylor's Theorem

If p is a n^{th} polynomial order polynomial with $p(a) = f(a), \ldots, p^{(n)}(a) = f^{(n)}(a)$ we can estimate the difference

$$f(x) - p(x).$$

Maximum possible error:

$$\frac{M}{(n+1)!}(x-a)^{n+1}$$

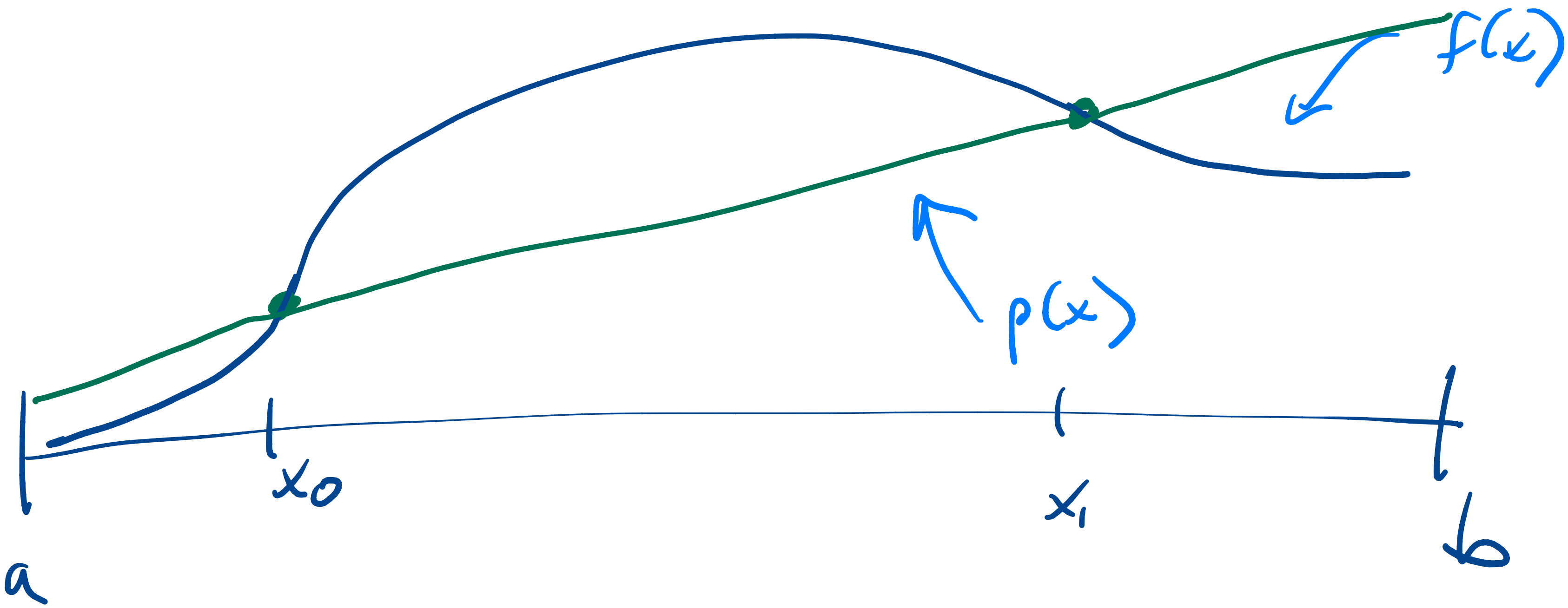
where

$$M = \max_{\xi \in [a,b]} |f^{(n+1)}(x)|.$$

Error in Polynomial Interpolation

Instead of doing a great job of approximating f and its derivatives at one point (a), we do a great job of approximating just the values of f at a bunch of points $x_0, \dots, x_n$. Now how big can the error be?

Error in Linear Interpolation



Error is zero if:

$x = x_0$

$x = x_1$

$f(x) - p(x)$

Error in Linear Interpolation

Error is zero if:

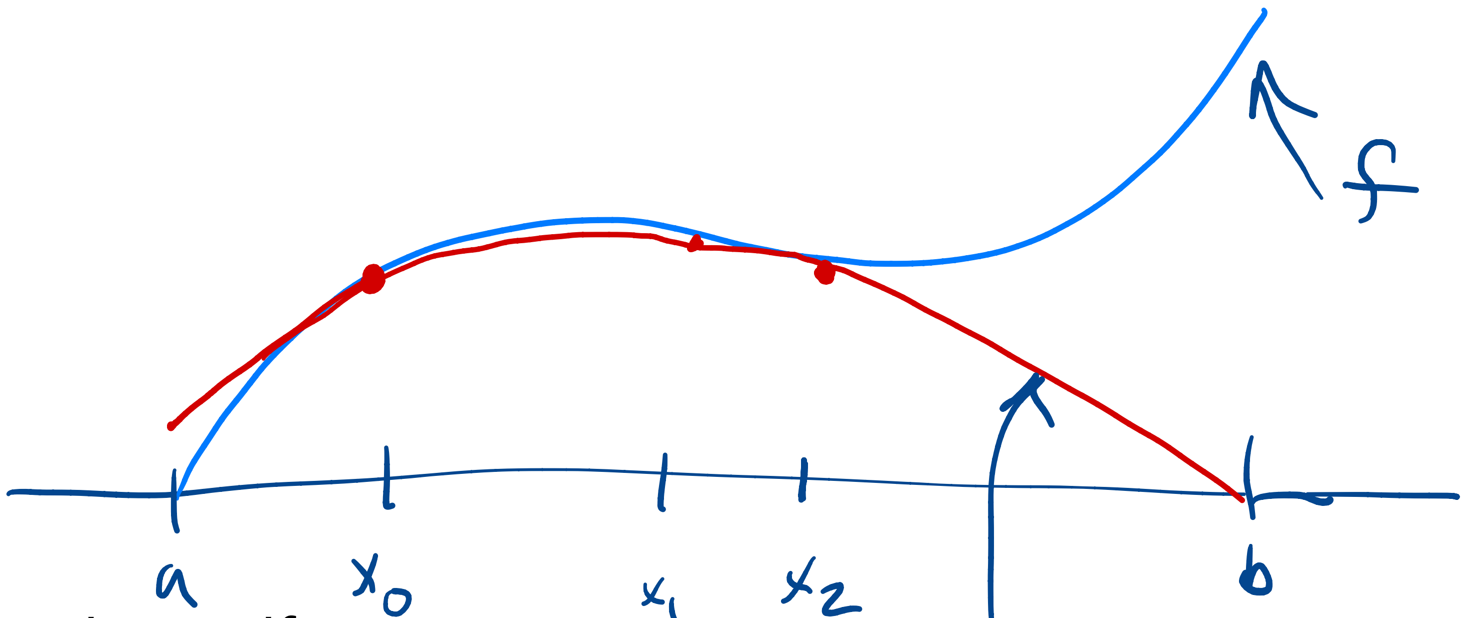
1. $x = x_0$
2. $x = x_1$

Error in Linear Interpolation

Error is zero if:

1. $x = x_0$
2. $x = x_1$
3. f is already linear $\left(f''(x) \equiv 0\right)$

Error in Quadratic Interpolation



Error is zero if:

$$f(x) - p(x)$$

$$x = x_0$$

$$x = x_1$$

$$x = x_2$$

$f'''(x) = 0$ everywhere

Error in Quadratic Interpolation

Error is zero if:

1. $x = x_0$
2. $x = x_1$
3. $x = x_2$

Error in Quadratic Interpolation

Error is zero if:

1. $x = x_0$
2. $x = x_1$
3. $x = x_2$
4. f is already quadratic $\left(f'''(x) \equiv 0\right)$

Error in Linear/Quadratic Interpolation

Linear case: Error should depend on

1. $x - x_0$
2. $x - x_1$
3. something about f''

Error in Linear/Quadratic Interpolation

Linear case: Error should depend on

1. $x - x_0$
2. $x - x_1$
3. something about f''

Quadratic case: Error should depend on

1. $x - x_0$
2. $x - x_1$
3. $x - x_2$
4. something about f'''

$$\boxed{\frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)}$$