Gaussian Quadrature

Math 426

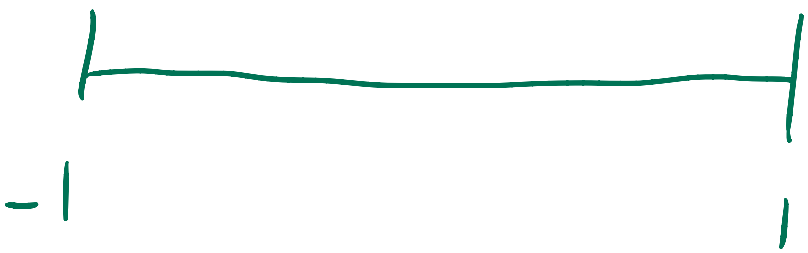
University of Alaska Fairbanks

November 18, 2020

Motivation

Suppose you want to integrate on $[-1,1]$ and I only let you have one sample point. Where do you put it so that $\int_{-1}^{1} p(x)\,dx$ is computed exactly for polynomials of as high a degree as possible?

$$A_0 f(x_0) \approx \int_{-1}^{1} f(x)\,dx$$



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$$\int_{-1}^{1} f(x) \approx A_0 f(x_0)$$

$$\frac{K f^{(k)}(\xi)}{k!}$$

Need to determine A_0 and x_0.

Motivation

Suppose you want to integrate on $[-1,1]$ and I only let you have one sample point. Where do you put it so that $\int_{-1}^{1} p(x)\,dx$ is computed exactly for polynomials of as high a degree as possible?

$$0 = 2x_0$$

$$\int_{-1}^{1} f(x) \approx A_0 f(x_0)$$

Need to determine A_0 and x_0.

$$\int_{-1}^{1} 1\,dx = A_0 1$$

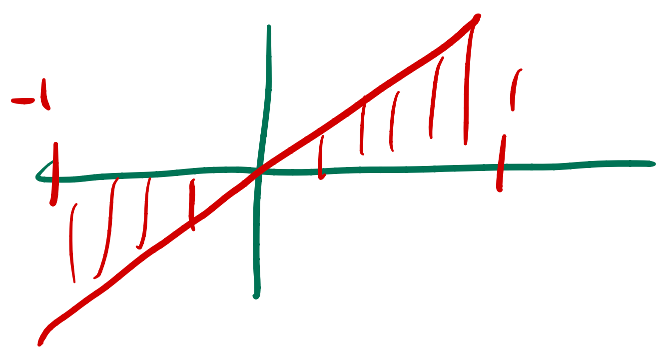
$$\int_{-1}^{1} 1\,dx = 2$$

$$f(x) \equiv 1 : \quad 2 = A_0$$

$$f(x) = x : \quad 0 = A_0 x_0$$

$$\int_{-1}^{1} x\,dx = A_0 x_0$$

$$\int_{-1}^{1} x\,dx = 0$$



Motivation

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$$\int_{-1}^{1} f(x) \approx A_0 f(x_0)$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3}$$

Need to determine A_0 and x_0.

$$2 \cdot 0 = 0$$

$$f(x) \equiv 1: \quad 2 = A_0$$
$$f(x) = x: \quad 0 = A_0 x_0$$

$A_0 = 2$

$x_0 = 0$

So put it in the midpoint: $x_0 = 0$ and $A_0 = 2$.

Two sample points, wah-ah-ah!

Suppose you want to integrate on $[-1,1]$ and I only let you have two sample points. Where do you put them so that $\int_{-1}^{1} p(x)\,dx$ is computed exactly for polynomials of as high a degree as possible?

$A_0, x_0 \qquad A_1, x_1$

Two sample points, wah-ah-ah!

Suppose you want to integrate on $[-1,1]$ and I only let you have two sample points. Where do you put them so that $\int_{-1}^{1} p(x)\,dx$ is computed exactly for polynomials of as high a degree as possible?

$$\int_{-1}^{1} f(x) \approx A_0 f(x_0) + A_1 f(x_1)$$

Need to determine four unknowns: A_0, A_1 and x_0, x_1.

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Need to determine four unknowns: A_0, A_1 and x_0, x_1.

$* A_0 + * A_1 + * x_0 + * x_1 = r_1$

$* A_0 + * A$

$$\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} r \end{bmatrix}$$

$$\begin{aligned}
f(x) \equiv 1 &: \quad 2 = A_0 + A_1 \quad r_2 \\
f(x) = x &: \quad 0 = A_0 x_0 + A_1 x_1 \\
f(x) = x^2 &: \quad \frac{2}{3} = A_0 x_0^2 + A_1 x_1^2 \\
f(x) = x^3 &: \quad 0 = A_0 x_0^3 + A_1 x_1^3
\end{aligned}$$

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$$\int_{-1}^{1} f(x) \approx A_0 f(x_0) + A_1 f(x_1)$$

Need to determine four unknowns: A_0, A_1 and x_0, x_1.

$$\begin{aligned} f(x) \equiv 1: &\quad 2 = A_0 + A_1 \\ f(x) = x: &\quad 0 = A_0 x_0 + A_1 x_1 \\ f(x) = x^2: &\quad \frac{2}{3} = A_0 x_0^2 + A_1 x_1^2 \\ f(x) = x^3: &\quad 0 = A_0 x_0^3 + A_1 x_1^3 \end{aligned}$$

This is a mess of nonlinear polynomial equations. Have fun.

Alternative Strategy

Dot product on polynomials:

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

$$[x_1, x_2] \cdot [y_1, y_2] \qquad x_1y_1 + x_2y_2$$

Alternative Strategy

$$\int_{-1}^{1} p_1(x)\, p_2(x)\, dx = 0$$

perpendicular

Dot product on polynomials:

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx.$$

Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

Alternative Strategy

Dot product on polynomials:

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

If $p(x)$ is a polynomial of degree 5 we can write

$q(x)\overline{)\,p(x)}$

$$p(x) = a(x)q(x) + b(x)$$

where $a(x)$ and $b(x)$ are polynomials of degree 2.

$$\begin{array}{r} 2 \\ 3\overline{)\,7} \\ \underline{6} \\ 1 \end{array}$$

2 R1

$7 = 2 \cdot 3 + 1$

Alternative Strategy

Dot product on polynomials:

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Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

If $p(x)$ is a polynomial of degree 5 we can write

$$p(x) = a(x)q(x) + b(x)$$

where $a(x)$ and $b(x)$ are polynomials of degree 2.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} a(x)q(x)\, dx + \int_{-1}^{1} b(x)\, dx = \int_{-1}^{1} b(x)\, dx$$

Alternative Strategy

Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

If $p(x)$ is a polynomial of degree 5 we can write → 3 roots

$$p(x) = a(x)q(x) + b(x)$$

where $a(x)$ and $b(x)$ are polynomials of degree 2.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} b(x)\, dx.$$

→ degree 2

x_k

1) Make $a(x)q(x)$ go away

A_k

2) do an exact job for poly's of degree 2.

Alternative Strategy

Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

If $p(x)$ is a polynomial of degree 5 we can write

$$p(x) = a(x)q(x) + b(x)$$

where $a(x)$ and $b(x)$ are polynomials of degree 2.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} b(x)\, dx.$$

Choose three sample points: x_0, x_1 and x_2, the roots of $q(x)$. Then choose weights A_k so that quadratics are integrated exactly.

Alternative Strategy

Suppose $q(x)$ is a polynomial of degree 3 and is perpendicular, in this sense, to all polynomials of degree 0, 1 and 2.

If $p(x)$ is a polynomial of degree 5 we can write

$$p(x) = a(x)q(x) + b(x)$$

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$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} b(x)\, dx.$$

Choose three sample points: x_0, x_1 and x_2, the roots of $q(x)$. Then choose weights A_k so that quadratics are integrated exactly.

$$\begin{aligned} Q[p] &= \sum_{k=0}^{2} A_k(a(x_k)\overset{=0}{q(x_k)} + b(x_k)) \\ &= \sum_{k=0}^{2} A_k b(x_k) = \int_{-1}^{1} b(x)\, dx = \int_{-1}^{1} p(x)\, dx \end{aligned}$$

Gaussian Quadrature

Suppose $q(x)$ is a polynomial of degree $n+1$ that is perpendicular to $1, x, \ldots, x^n$.

$(n=2)$

$n+1$

Gaussian Quadrature

Suppose $q(x)$ is a polynomial of degree $n+1$ that is perpendicular to $1, x, \ldots, x^n$.

$n = 2$

Let $x_0, \ldots x_n$ be the roots of q. Let $A_0, \ldots, A_n$ be weights so that Q is exact for polynomials of degree n.

5 sample points

exact for

degree ≤ 4

$\underbrace{0, 1, 2, 3, 4}$

Gaussian Quadrature

Suppose $q(x)$ is a polynomial of degree $n+1$ that is perpendicular to $1, x, \ldots, x^n$.

Let $x_0, \ldots x_n$ be the roots of q. Let $A_0, \ldots, A_n$ be weights so that Q is exact for polynomials of degree n.

$n = 2$

If $p(x)$ is a polynomial of degree $2n+1$,

$$p(x) = \underbrace{a(x)}_{n} q(x) + b(x)$$

with $a(x)$, $b(x)$ polynomials of degree n.

$$\int_{-1}^{1} p(x) = \int_{-1}^{1} a(x)q(x) + b(x)\,dx = \int_{-1}^{1} b(x)\,dx$$

Gaussian Quadrature

Suppose $q(x)$ is a polynomial of degree $n+1$ that is perpendicular to $1, x, \ldots, x^n$.

Let $x_0, \ldots x_n$ be the roots of q. Let $A_0, \ldots, A_n$ be weights so that Q is exact for polynomials of degree n.

If $p(x)$ is a polynomial of degree $2n+1$,

$$p(x) = a(x)q(x) + b(x)$$

with $a(x)$, $b(x)$ polynomials of degree n.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} b(x)\, dx$$

Gaussian Quadrature

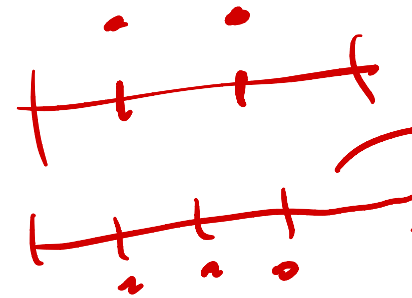
Suppose $q(x)$ is a polynomial of degree $n+1$ that is perpendicular to $1, x, \ldots, x^n$.

$n+1 \to$ degree n

Let $x_0, \ldots x_n$ be the roots of q. Let $A_0, \ldots, A_n$ be weights so that Q is exact for polynomials of degree n.

$\to$ degree 1

If $p(x)$ is a polynomial of degree $2n+1$,



$$p(x) = a(x)q(x) + b(x)$$

$\to$ degree 2

with $a(x)$, $b(x)$ polynomials of degree n.

$$\int_{-1}^{1} p(x)\, dx = \int_{-1}^{1} b(x)\, dx$$

of degree $2n+1$

$$Q[p] = \sum_{k=0}^{n} A_k(a(x_k)q(x_k) + b(x_k))$$

$$= \sum_{k=0}^{n} A_k b(x_k) = \int_{-1}^{1} b(x)\, dx = \int_{-1}^{1} p(x)\, dx.$$

Can we find a quadratic that is perpendicular to poly'ls of degree 0, 1 ?

q

$$q(x) = x^2 - \alpha_1 x - \alpha_0 1$$

Gaussain Quadrature

How do we find a quadratic polynomial that is perpendicular to 1 and x?

Gramm-Schmidt!

1   x

perp.

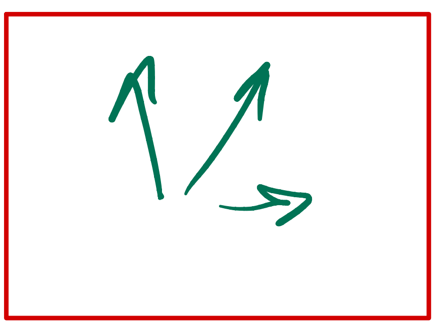
$\tilde{q}_0(x) = 1$

How to compute a linear $\tilde{q}_1$ that is perpendicular to $\tilde{q}_0$?

$$\int_{-1}^{1} 1 \cdot x \, dx = 0$$

$$\int_{-1}^{1} \tilde{q}_0(x)(x - \alpha_0 1) = 0$$

So $\alpha_0 = 0$.



$$\int_{-1}^{1} 1(x - \alpha_0 1)\,dx = 0 - \alpha_0 2$$

$$0 = -\alpha_0 2$$

$$\alpha_0 = 0$$

Gaussain Quadrature

How do we find a quadratic polynomial that is perpendicular to 1 and x?

$\tilde{q}_0 = 1$

$\tilde{q}_1 = x$

Gramm-Schmidt!

$\tilde{q}_0(x) = 1$

How to compute a linear $\tilde{q}_1$ that is perpendicular to $\tilde{q}_0$?

$$\int_{-1}^{1} q_0(x)(x - \alpha_0 1) = 0$$

So $\alpha_0 = 0$.

How to compute a quadratic $\tilde{q}_2$ that is perpendicular to $\tilde{q}_1$ and $\tilde{q}_0$?

$$\int_{-1}^{1} \tilde{q}_0(x)(x^2 - (\alpha_0 \tilde{q}_0 + \alpha_1 \tilde{q}_1)\ dx = 0$$

$$\int_{-1}^{1} \tilde{q}_1(x)(x^2 - (\alpha_0 \tilde{q}_0 + \alpha_1 \tilde{q}_1)\ dx = 0$$

$\to 0$

$\int_{-1}^{1} \tilde{q}_0(x)\tilde{q}_1(x)\,dx$

Roots and Weights

$$\int_{-1}^{1} \tilde{q}_0(x)x^2\,dx = \alpha_0 \int_{-1}^{1} (\tilde{q}_0)^2\,dx$$

$$\int_{-1}^{1} \tilde{q}_1(x)x^2\,dx = \alpha_1 \int_{-1}^{1} (\tilde{q}_1)^2\,dx$$

$\tilde{q}_0(x) = 1$

$\tilde{q}_1(x) = x$

$$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = \alpha_0 \int_{-1}^{1} 1\,dx = 2\alpha_0$$

$$0 = \int_{-1}^{1} x^3\,dx = \alpha_1 \int_{-1}^{1} x^2\,dx = \frac{2}{3}\alpha_1$$

$\alpha_1 = 0$

$\alpha_0 = 1/3$

Roots and Weights

$$\int_{-1}^{1} \tilde{q}_0(x)x^2\,dx = \alpha_0 \int_{-1}^{1} (\tilde{q}_0)^2\,dx$$
$$\int_{-1}^{1} \tilde{q}_0(x)x^2\,dx = \alpha_1 \int_{-1}^{1} (\tilde{q}_1)^2\,dx$$

$$\int_{-1}^{1} x^2\,dx = \alpha_0 \int_{-1}^{1} 1\,dx$$
$$\int_{-1}^{1} x \cdot x^2\,dx = \alpha_1 \int_{-1}^{1} x^2\,dx$$

Roots and Weights

$$\int_{-1}^{1} \tilde{q}_0(x)x^2\,dx = \alpha_0 \int_{-1}^{1} (\tilde{q}_0)^2\,dx$$
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$$\frac{2}{3} = 2\alpha_0$$
$$0 = \alpha_1 \frac{2}{3}$$

Roots and Weights

$$\frac{2}{3} = 2\alpha_0$$
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Roots and Weights

$$\frac{2}{3} = 2\alpha_0$$

$$0 = \alpha_1 \frac{2}{3}$$

$$\boxed{q(x) = x^2 - \frac{1}{3}}$$

$$x^2 = \alpha_0 \tilde{q}_0 - \alpha_1 \tilde{q}_1$$

$$\frac{1}{3} \quad 1 \quad 0 \quad x$$

$q \perp 1$, $q \perp x$

sample points: $q(x) = 0$

$$x^2 - \frac{1}{3} = 0 \qquad x = \pm \frac{1}{\sqrt{3}}$$

Roots and Weights

$$\frac{2}{3} = 2\alpha_0$$
$$0 = \alpha_1 \frac{2}{3}$$

$$q(x) = x^2 - \frac{1}{3}$$

$$x_0 = -\frac{1}{\sqrt{3}}; \qquad x_1 = \frac{1}{\sqrt{3}}$$

Roots and Weights

$$\frac{2}{3} = 2\alpha_0$$
$$0 = \alpha_1 \frac{2}{3}$$

$$q(x) = x^2 - \frac{1}{3}$$

$$x_0 = -\frac{1}{\sqrt{3}}; \qquad x_1 = \frac{1}{\sqrt{3}}$$

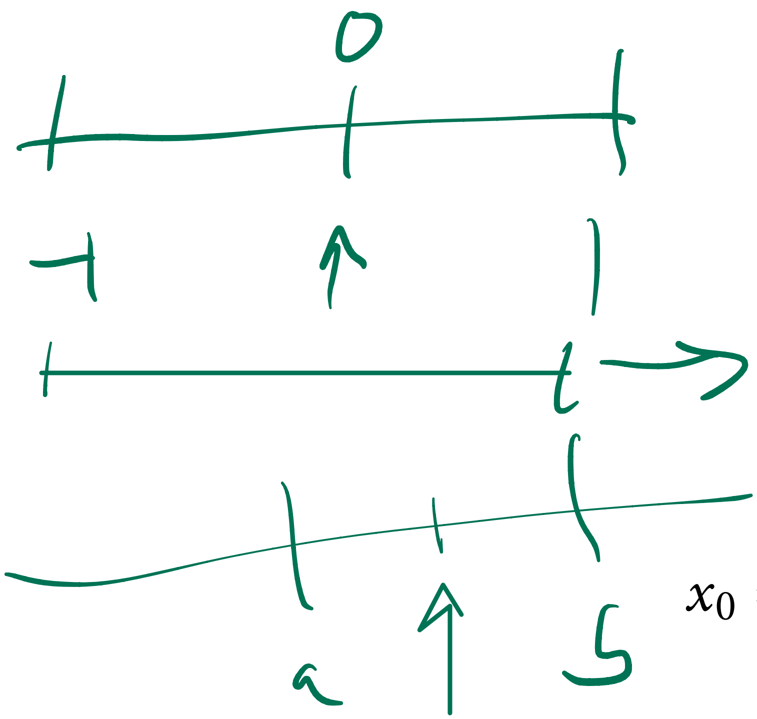
Now pick weights A_0 and A_1 so that integration of linear functions is exact.

$$A_0 + A_1 = 2$$
$$A_0 \frac{-1}{\sqrt{3}} + A_1 \frac{1}{\sqrt{3}} = 0$$

$A_0 + A_1 = 2$

$(A_0 - A_1)\frac{1}{\sqrt{3}} = 0$

Roots and Weights



$$\frac{2}{3} = 2\alpha_0$$

$$0 = \alpha_1 \frac{2}{3}$$

$$q(x) = x^2 - \frac{1}{3}$$

$$x_0 = -\frac{1}{\sqrt{3}}; \qquad x_1 = \frac{1}{\sqrt{3}}$$

$$\boxed{x_0 = \frac{-1}{\sqrt{3}} \quad x_1 = \frac{1}{\sqrt{3}} \quad A_0 = 1 \quad A_1 = 1}$$

exact job for cubics

$(2\cdot 1+1)$

Now pick weights A_0 and A_1 so that integration of linear functions is exact.

$m \quad \int_a^b (x-m)^3 = 0$

$$A_0 + A_1 = 2$$

$$A_0 \frac{-1}{\sqrt{3}} + A_1 \frac{1}{\sqrt{3}} = 0$$

$(x-m)^k$

$2(x-7)+15$

$m = 7$

$2x+1 = 2(x-7)+2\cdot 7+1$

$A_0 = 1$ and $A_1 = 1$.