Quadrature Review

Math 426

University of Alaska Fairbanks

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Formulas from Polynomial Interpolation

Start with $f(x)$ on $[a, b]$.

$$\int_a^b f(x)\,dx$$

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$\int_a^b f(x) dx$

$$Q[f] = \int_a^b p(x)$$

$f(x) \approx p(x)$

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When the sample points are equally spaced and include a and b, these are called Newton-Coates formulas.

$n=1$: Trapezoid Rule

$n=2$: Simpson's Rule

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$\xi \in [a,b]$

Newton-Coates error:

$$\int_a^b f(x)\,dx - Q[f] = \begin{cases} K f^{(n+1)}(\xi) & n \text{ odd} \\ K f^{(n+2)}(\xi) & n \text{ even} \end{cases}$$

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Simpson's Rule: $n = 2$, $K = -\frac{1}{90}(h/2)^5$

$-\frac{1}{90}\left(\frac{h}{2}\right)^5 f^{(4)}(\xi)$

Determining Quadrature Formulas

If $x_0, \ldots, x_n$ are the sample points,

$$Q[f] = A_0 f(x_0) + \cdots + A_n f(x_n)$$

where

$$A_k = \int_a^b \phi_k(x)\,dx.$$

Lagrange interpolating polynomials

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n+1 A's

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n+1 equations

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$1, x^2, 1+x^2$

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$$\int_a^b q(x)\, dx = Q[q] = A_0 q(x_0) + \cdots + A_n q(x_n)$$

Pick $n+1$ linearly independent polynomials q_k of degree n or less to determine $n+1$ equations to solve for the A_k's.

$$\phi_0(x)\, f(x_0) + \cdots + \phi_n(x) f(x_n)$$

$p(x)$ $\qquad p(x_k) = f(x_k) \quad \forall x_k$

$$\int_a^b p(x)\,dx = f(x_0) \int_a^b \phi_0(x)\,dx$$

$$+ \cdots + f(x_n) \int_a^b \phi_n(x)\,dx$$

$$= A_0 f(x_0) + \cdots + A_n f(x_n)$$

cubic $\qquad$ quartic interpolating

$f \qquad\qquad p$

$$\int_a^b f(x)\,dx = \int_a^b p(x)\,dx = Q[f]$$

$$= A_0 f(x_0) + \cdots + A_n f(x_n)$$