Cubic Spline Interpolation

For $2 \le k \le n-1$

$$\beta_k = \frac{\Delta x_k}{6}; \quad \alpha_{k-1} = \frac{1}{3}(\Delta x_{k-1} + \Delta x_k)$$

$$\beta_{k-1} z_{k-2} + \alpha_{k-1} z_{\underset{k-1}{\cancel{k}}} + \beta_k z_{\underset{k}{\cancel{k+1}}} = \Delta\Delta f_{\underset{k-1}{\cancel{k}}}$$

$$\begin{array}{cccccc} 1 & 0 & & & & 0 \end{array}$$

$$\begin{pmatrix} * & \cdots & \cdots & \cdots & \cdots & * \\ \beta_1 & \alpha_1 & \beta_2 & 0 & 0 & \cdots \\ 0 & \beta_2 & \alpha_2 & \beta_3 & 0 & \cdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \beta_{n-1} & \alpha_{n-1} & \beta_n \\ * & \cdots & \cdots & \cdots & \cdots & * \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{pmatrix} = \begin{pmatrix} * \\ \Delta\Delta f_1 \\ \vdots \\ \Delta\Delta f_{n-1} \\ * \end{pmatrix}$$

$$\begin{array}{cccccc} 0 & & & & 0 & 1 \end{array}$$

Upshot: If the spline has continuous first and second derivatives at the interior nodes, then the second derivatives z_k satisfy the above system. This almost determines the z_k's up to two additional conditoins that you get to specify.

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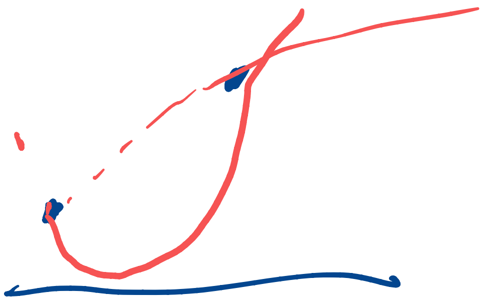

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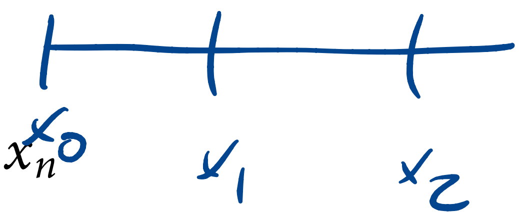

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Not-a knot Third derivatives exist at x_1, x_{n-1}. No information explict at x_0 or x_1. On interval k,

$$s'''(x) = \frac{z_k - z_{k-1}}{\Delta x_k}$$

$$\frac{z_1 - z_0}{\Delta x_1} = \frac{z_2 - z_1}{\Delta x_2}$$

which gives a condition on z_0, z_1 and z_2.